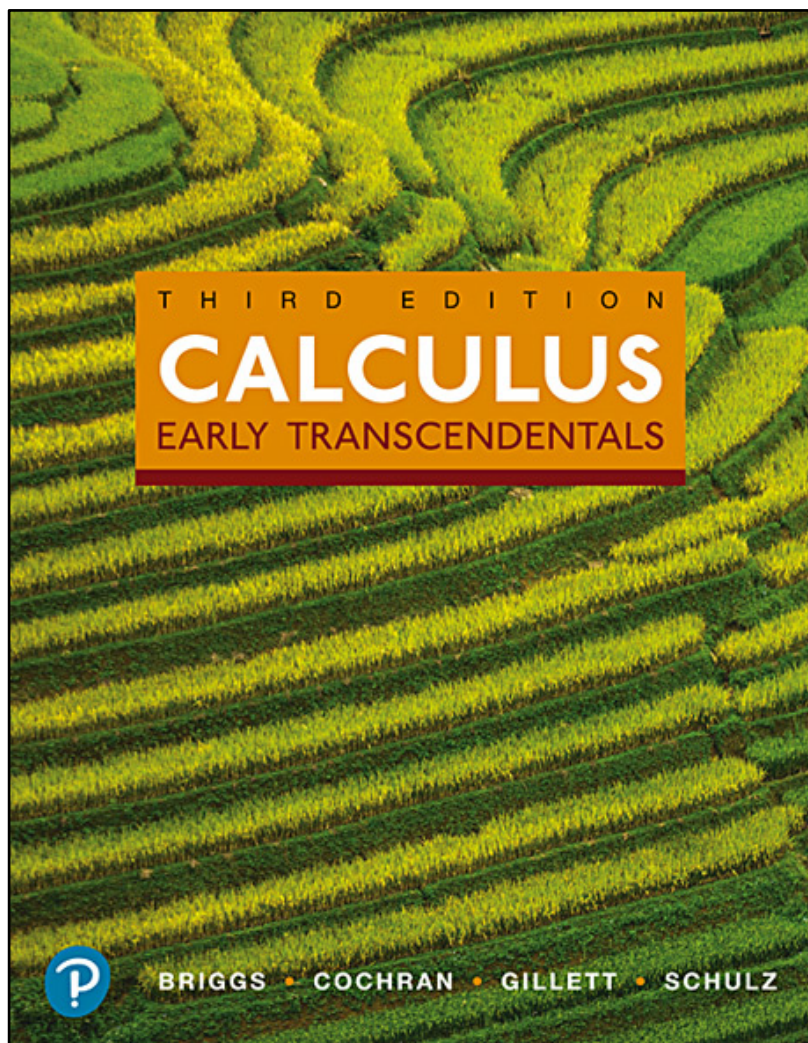


Calculus Early Transcendentals

Third Edition



Chapter 3 Derivatives

12 Lectures

3.1, 3.2, 3.3-3.4, 3.5, 3.6-3.7,
3.8, 3.9, 3.9, 3.10, 3.11, 3.11

Section 3.1 Introducing the Derivative

HW # 1, 3, 5, 13, 15, 25, 29, 31, 41, 44, 47, 51, 52, 60

Topics

- definition of derivative
- computing derivative through definition
- computing an equation of the line tangent to a graph $y = f(x)$.

$s(t)$

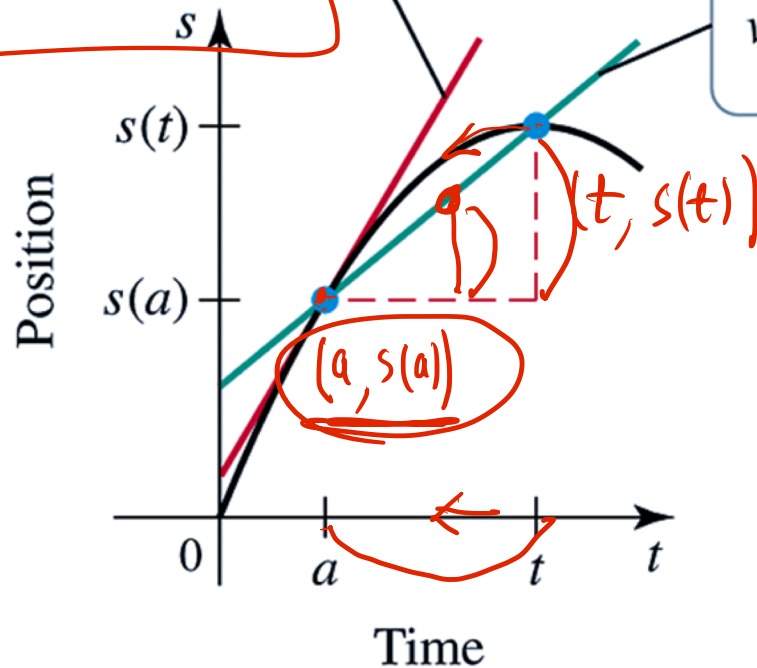
Figure 3.3

$$v_{\text{inst}} = m_{\text{tan}} = \lim_{t \rightarrow a} \frac{s(t) - s(a)}{t - a}$$

$$= s'(a) = \frac{ds(t)}{dt} = \frac{ds}{dt} \Big|_{t=a}$$

$$y - s(a) = m_{\text{tan}} (t - a)$$

$$v_{\text{av}} = m_{\text{sec}} = \frac{s(t) - s(a)}{t - a}$$



$$m_{\text{sec}} = \frac{y - s(a)}{t - a}$$

Example 1 a rock moving position $s(t) = -16t^2 + 96t$. Find

Figure 3.4

(a) instantaneous velocity at $t = 1$

(b) equation of the tangent line at $(1, 80)$

$$s'(1) = \lim_{t \rightarrow 1} \frac{s(t) - s(1)}{t - 1} \rightarrow 0$$

$$= \lim_{t \rightarrow 1} \frac{(-16t^2 + 96t) - (-16 + 96)}{t - 1}$$

$$= \lim_{t \rightarrow 1} \frac{-16(t^2 - 1) + 96(t - 1)}{t - 1}$$

$$= \lim_{t \rightarrow 1} [-16(t+1) + 96]$$

$$= -16 \cdot 2 + 96 = 64$$

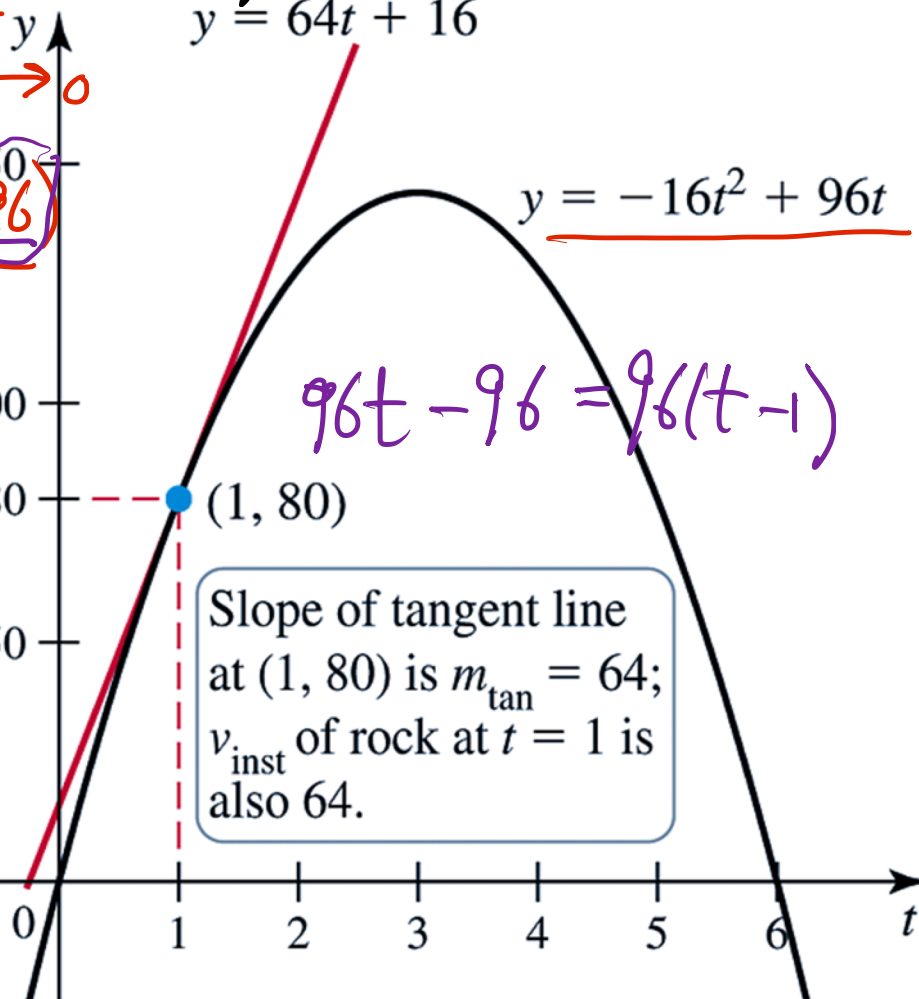
$$t^2 - 1 = (t-1)(t+1)$$

$$y = 64t + 16$$

$$y = -16t^2 + 96t$$

$$96t - 96 = 96(t - 1)$$

$$y - 80 = 64(t - 1)$$



Slope of tangent line
at $(1, 80)$ is $m_{\text{tan}} = 64$;
 v_{inst} of rock at $t = 1$ is
also 64.

Figure 3.5

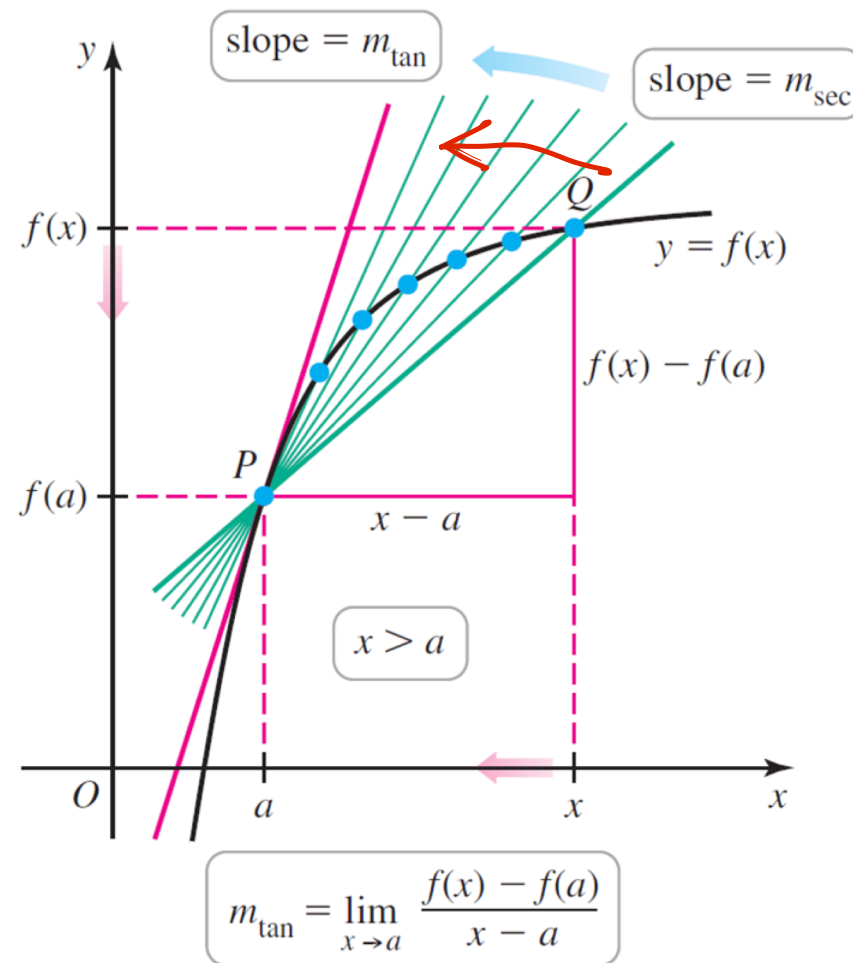
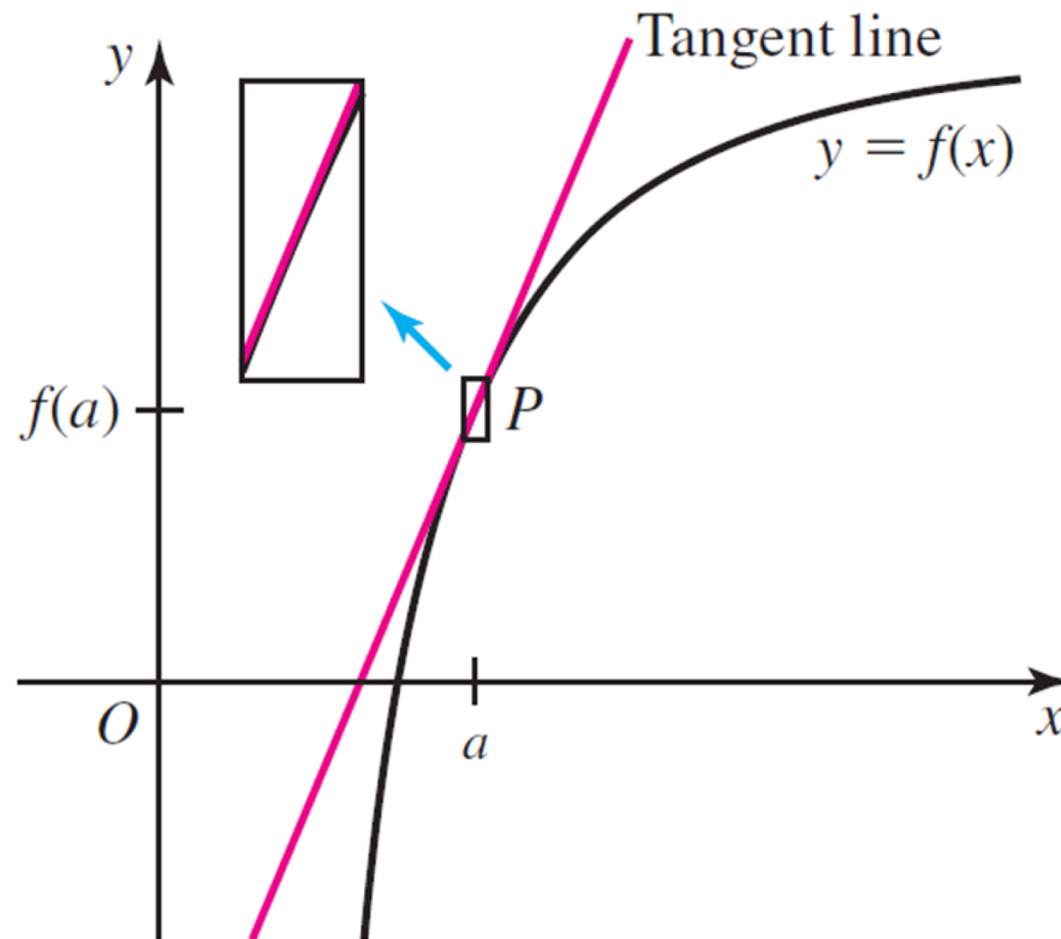


Figure 3.6



Definition Rate of Change and the Slope of the Tangent Line

The **average rate of change** in f on the interval $[a, x]$ is the slope of the corresponding secant line:

$$m_{\text{sec}} = \frac{f(x) - f(a)}{x - a}.$$

The **instantaneous rate of change** in f at a is

$$m_{\text{tan}} = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}, \quad (1)$$

which is also the **slope of the tangent line** at $(a, f(a))$, provided this limit exists.

The **tangent line** is the unique line through $(a, f(a))$ with slope m_{tan} .

Its equation is

$$y - f(a) = m_{\text{tan}}(x - a).$$

Definition The Derivative of a Function at a Point

The **derivative of f at a** , denoted $f'(a)$, is given by either of the two following limits, provided the limits exist and a is in the domain of f :

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \quad (1) \quad \text{or} \quad f'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}. \quad (2)$$

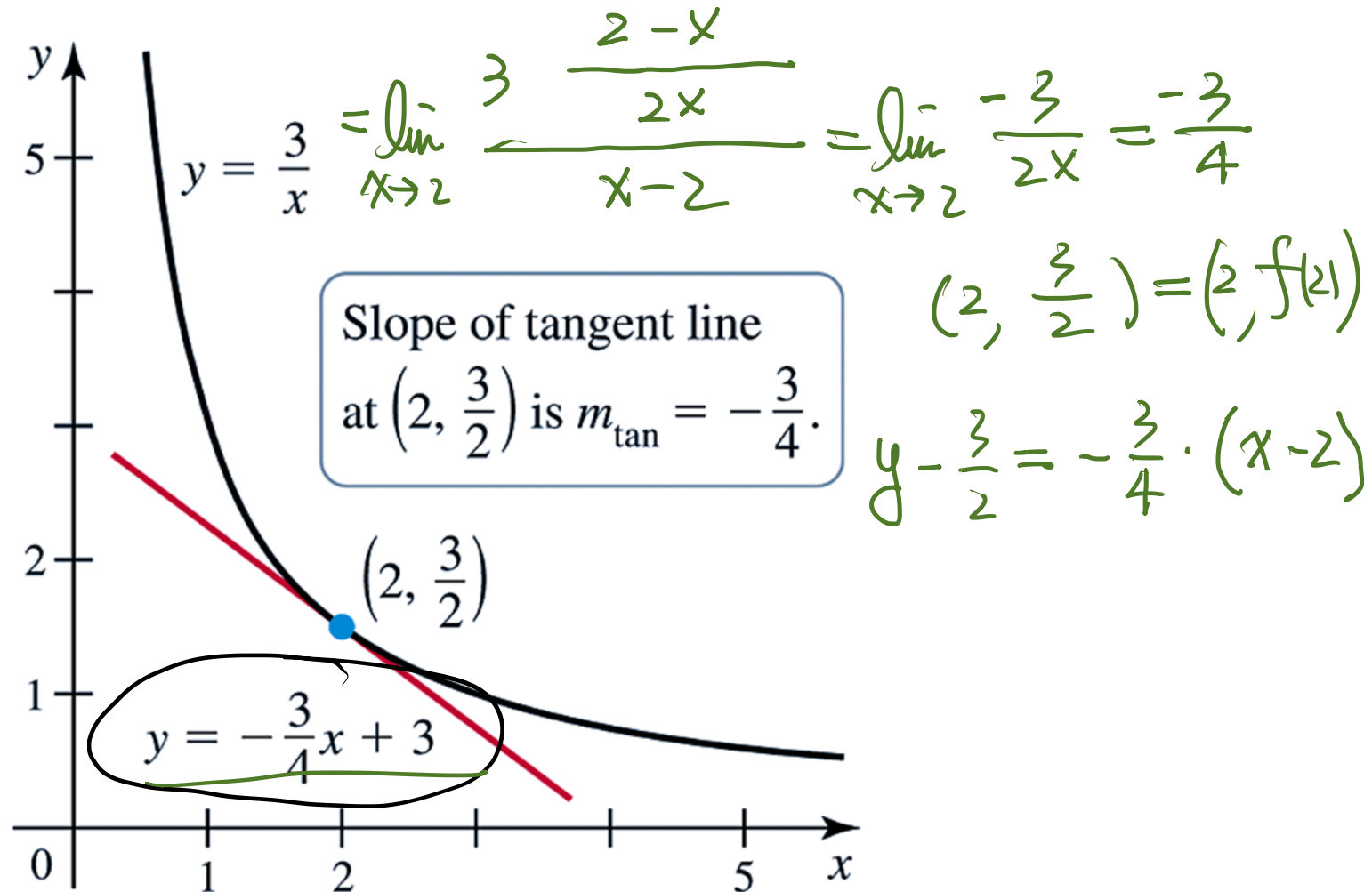
If $f'(a)$ exists, we say that f is differentiable at a .

$$h = x - a \rightarrow 0$$

Example 2 Find an equation of the line tangent to the graph of

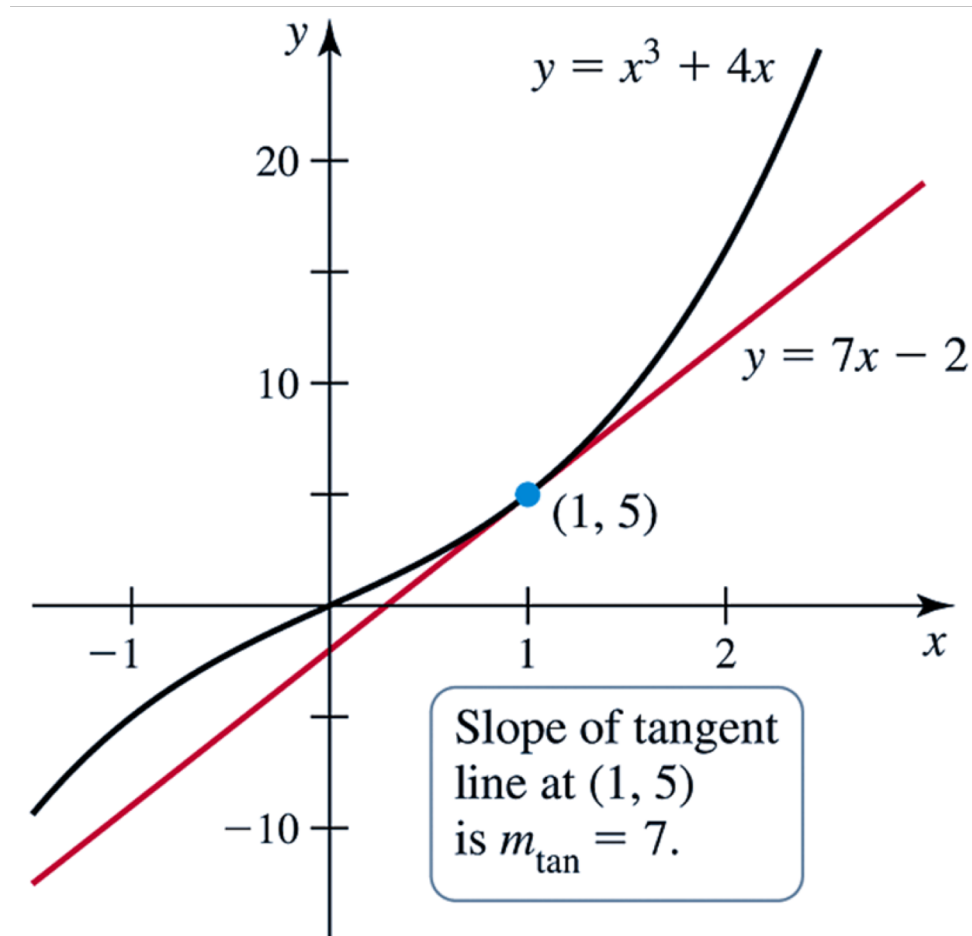
$$f(x) = \frac{3}{x} \text{ at } (2, \frac{3}{2})$$

Figure 3.7 $f'(2) = \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2} \frac{\frac{3}{x} - \frac{3}{2}}{x - 2} \rightarrow 0$



Example 3 Find an equation of the line tangent to the graph of $f(x) = x^3 + 4x$ at $(1, 5)$.

Figure 3.9



Example 4 $f(x) = \sqrt{2x+1}$. Compute $f'(2)$ and an eq. of the tangent line at $(2, 3)$.

Figure 3.10

$$a^2 - b^2 = (a-b)(a+b)$$

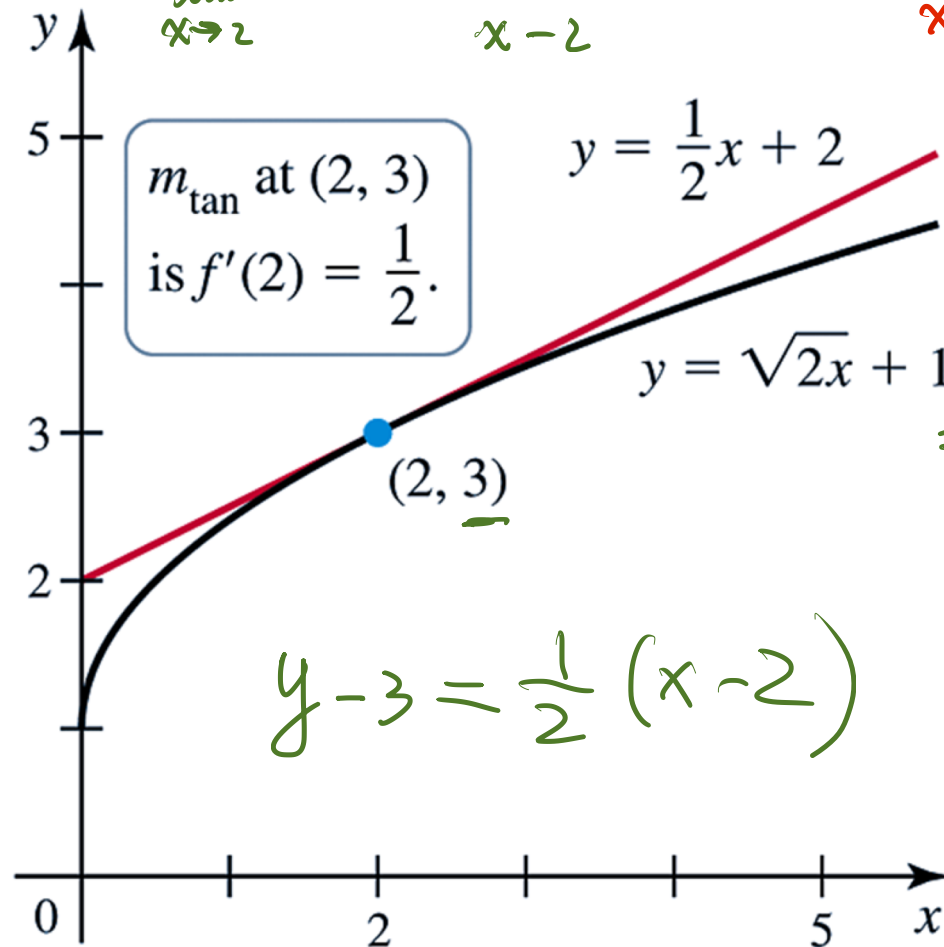
$$f'(2) = \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2}$$

$$= \lim_{x \rightarrow 2} \frac{[\sqrt{2x+1}] - [\sqrt{2 \cdot 2 + 1}]}{x - 2}$$

$$= \lim_{x \rightarrow 2} \frac{\sqrt{2x} - \sqrt{4}}{x - 2} \cdot \frac{\sqrt{2x} + \sqrt{4}}{\sqrt{2x} + \sqrt{4}}$$

$$= \lim_{x \rightarrow 2} \frac{2x - 4}{(x-2)(\sqrt{2x} + 2)} = \lim_{x \rightarrow 2} \frac{2(x-2)}{(x-2)(\sqrt{2x} + 2)}$$

$$= \lim_{x \rightarrow 2} \frac{2}{\sqrt{2x} + 2} = \frac{2}{2+2} = \frac{1}{2}$$



m_{tan} at $(2, 3)$
is $f'(2) = \frac{1}{2}$.

$$y = \frac{1}{2}x + 2$$

$$y = \sqrt{2x+1}$$

$(2, 3)$

$$y - 3 = \frac{1}{2}(x - 2)$$

Example Given $f'(3) = 2$, compute

$$\lim_{h \rightarrow 0} \frac{f(3+4h) - f(3-7h)}{5h}$$

$$(3+4h) - (3-7h) = 11h$$

$$\Rightarrow 2 = f'(3) = \lim_{h \rightarrow 0} \frac{f(3+4h) - f(3-7h)}{11h}$$

$$= \lim_{h \rightarrow 0} \frac{f(3+4h) - f(3-7h)}{5h} \cdot \frac{5}{11}$$

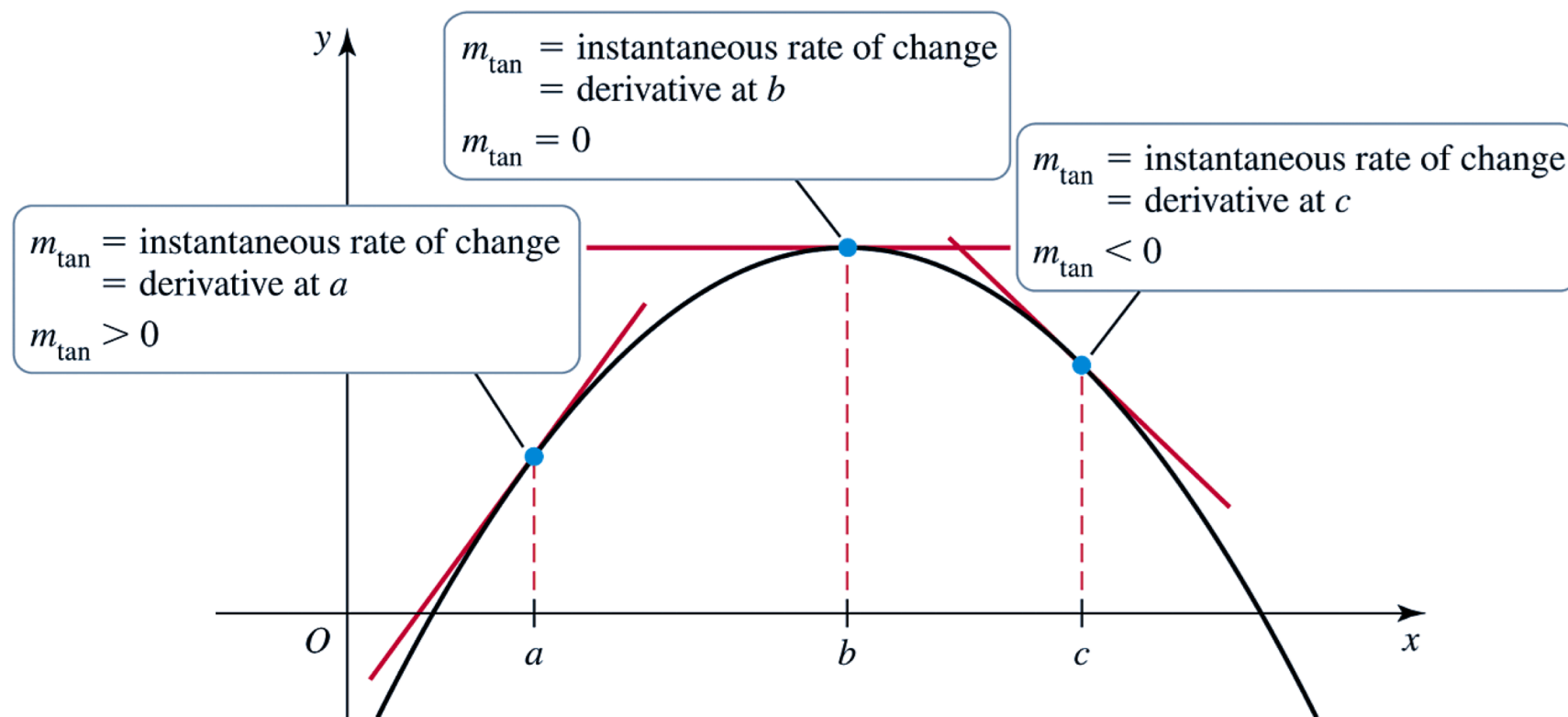
$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(3+4h) - f(3-7h)}{5h} = 2 \cdot \frac{5}{11} = \frac{10}{11}$$

Section 3.2 The Derivative as a Function

HW # 7, 8, 15, 17, 19, 35, 37, 39, 41, 43, 45, 51, 54, 55, 57, 60, 62, 66, 69, 73, 77.

- derivative of a function at $\begin{cases} \text{a fixed point } \underline{a} \\ \text{arbitrary point } x \end{cases}$
- continuity vs differentiability

Figure 3.14



Definition The Derivative Function

The **derivative** of f is the function

$$f'(\underline{x}) = \lim_{h \rightarrow 0} \frac{f(\underline{x+h}) - f(\underline{x})}{h},$$

provided the limit exists and x is in the domain of f . If $f'(x)$ exists, we say that f is **differentiable** at x . If f is differentiable at every point of an open interval I , we say that f is differentiable on I .

Example 1

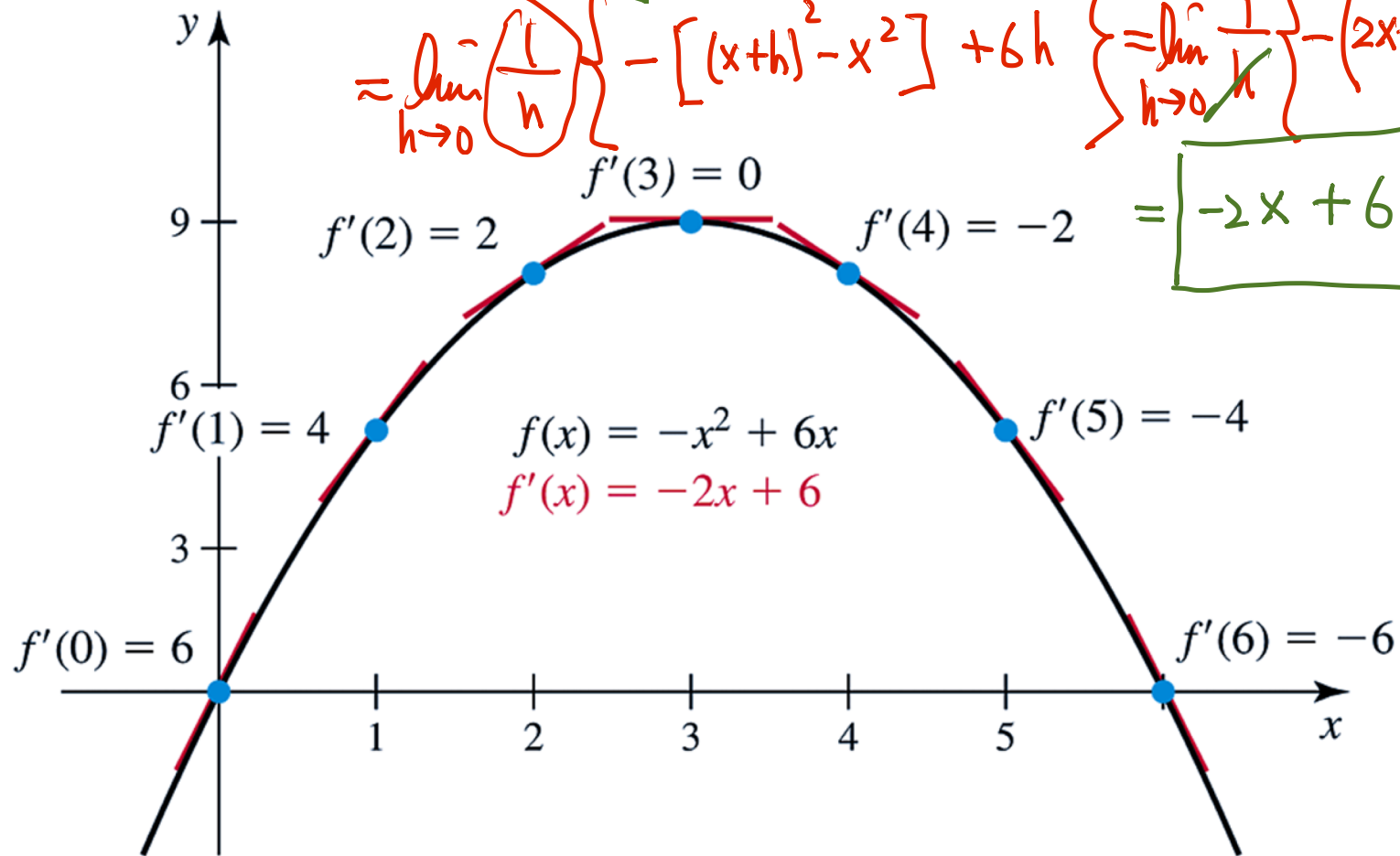
$f(x) = -x^2 + 6x$, compute $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{(x+h) - x}$

Figure 3.15

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left\{ \underbrace{-(x+h)^2}_{\text{red}} + \underbrace{6(x+h)}_{\text{red}} - \underbrace{(-x^2 + 6x)}_{\text{red}} \right\}$$

$$\approx \lim_{h \rightarrow 0} \left(\frac{1}{h} \right) \left\{ -[(x+h)^2 - x^2] + 6h \right\} = \lim_{h \rightarrow 0} \frac{1}{h} \left\{ -(2x+h) + 6h \right\}$$

$$= \boxed{-2x + 6 = f'(x)}$$



Example 2

$$f(x) = \sqrt{x}$$

Figure 3.17

$$(a-b)(a+b) = a^2 - b^2$$

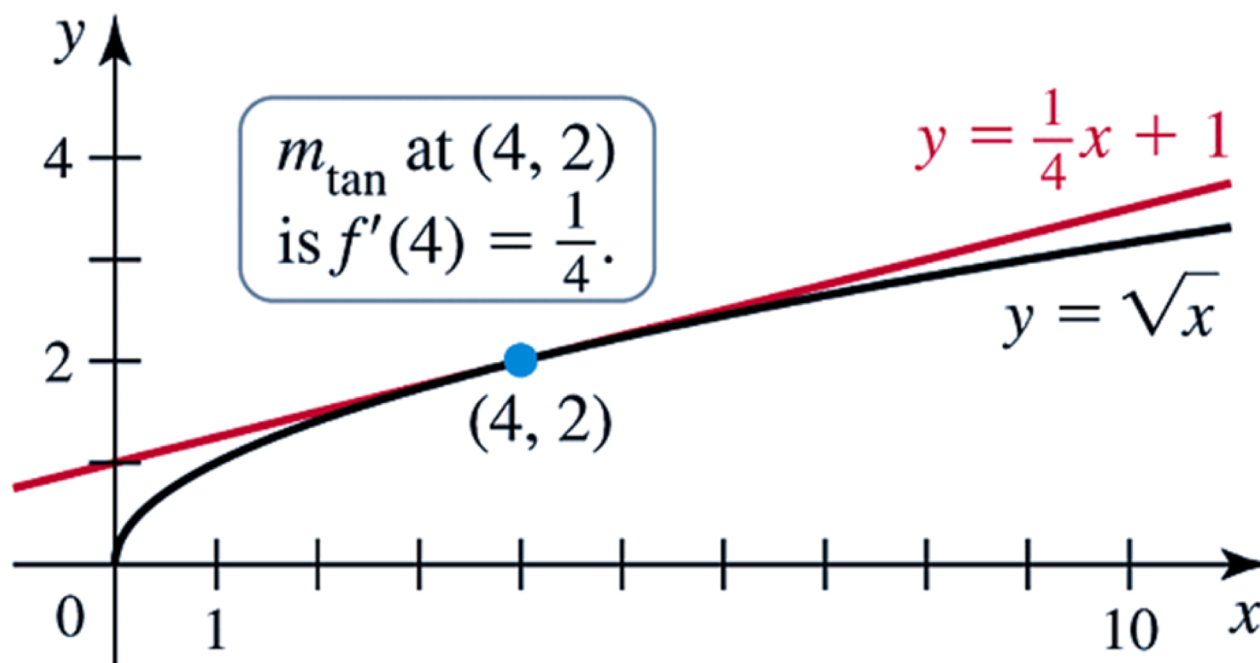
- compute $f'(x) = \lim_{h \rightarrow 0} \frac{1}{h} [\sqrt{x+h} - \sqrt{x}] \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$
- domains of f and f'

$$[0, +\infty)$$

$$= \lim_{h \rightarrow 0} \frac{1}{\cancel{h}} \cdot \frac{\cancel{h}}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}} = f'(x)$$

$$[0, +\infty)$$

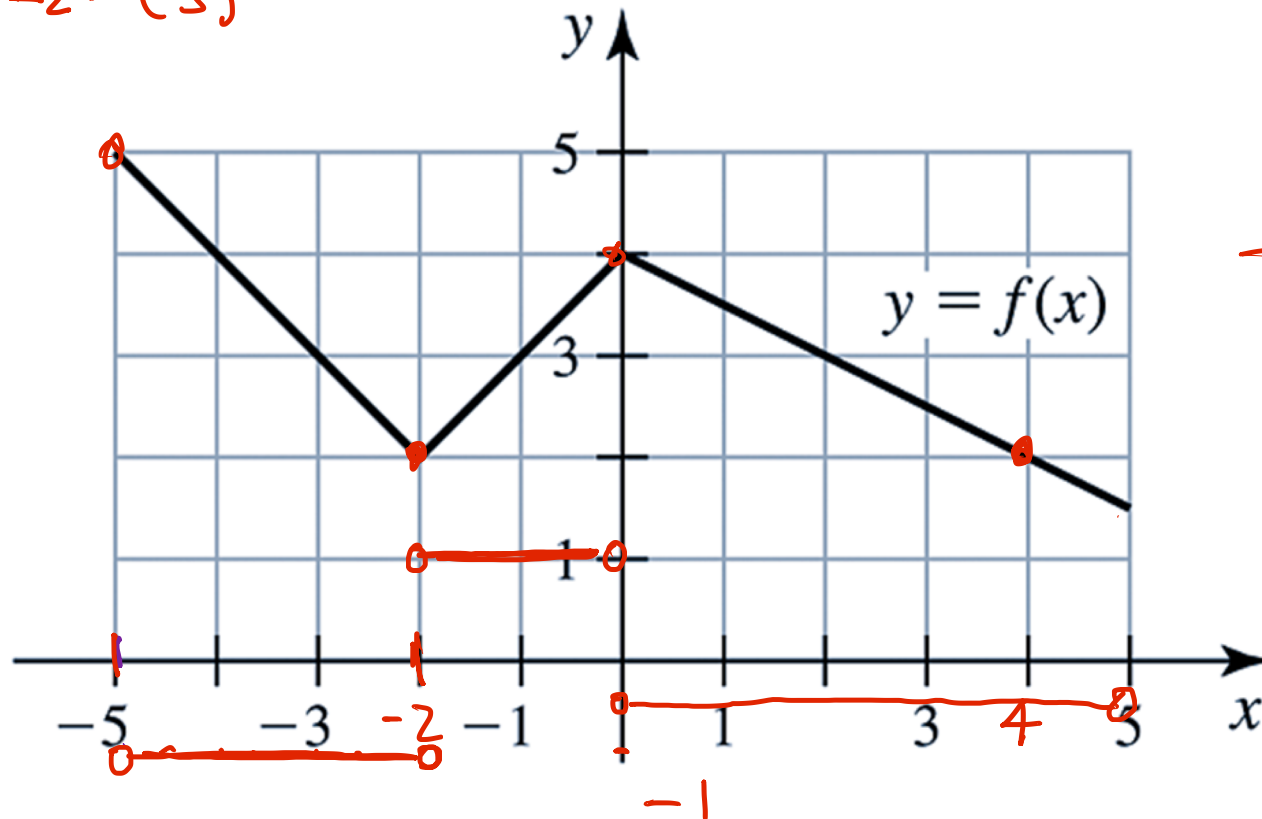


Example 4 sketch the graph of $f'(x)$.

Figure 3.18

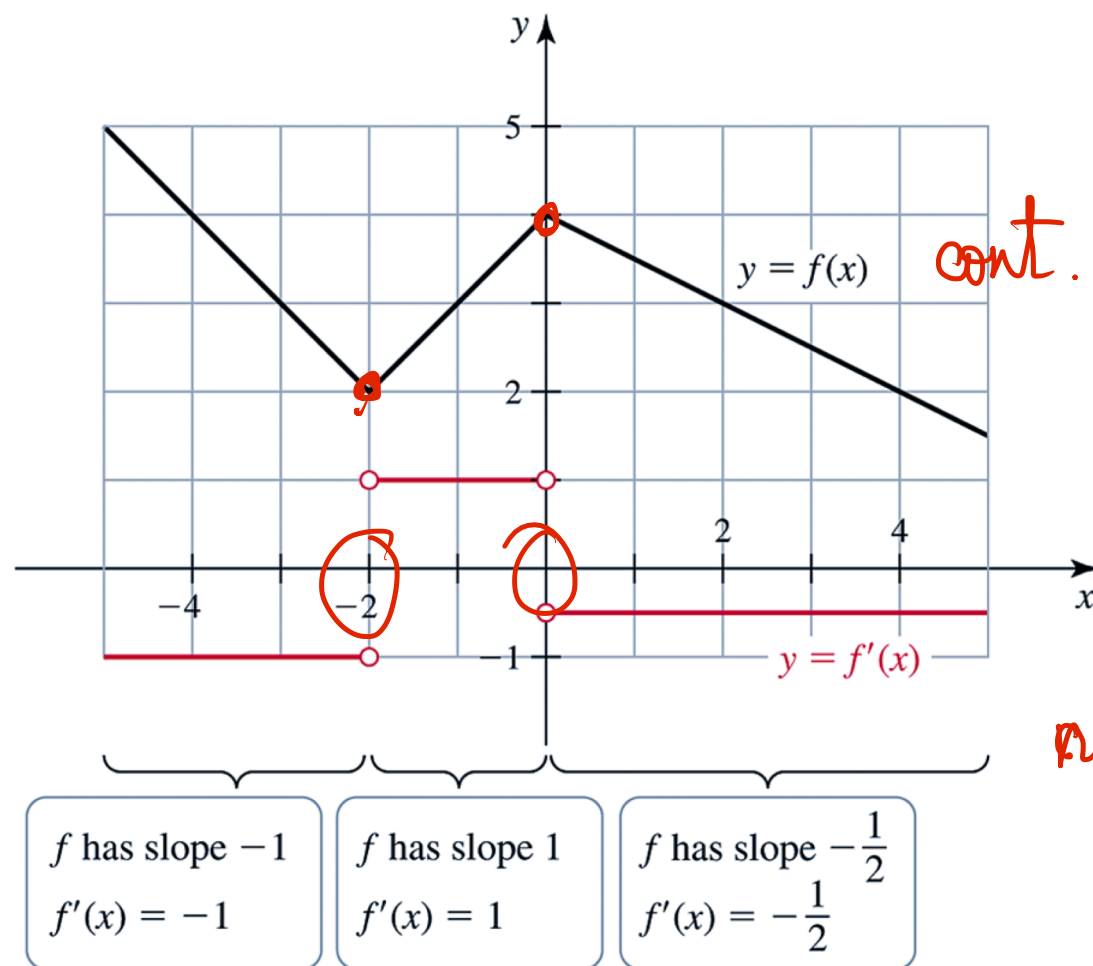
$$\frac{2-5}{-2-(-5)} = \frac{-3}{3} = -1$$

$$\frac{4-2}{0-(-2)} = \frac{2}{2} = 1$$



$$\frac{2-4}{4-0} = -\frac{2}{4} = -\frac{1}{2}$$

Figure 3.19

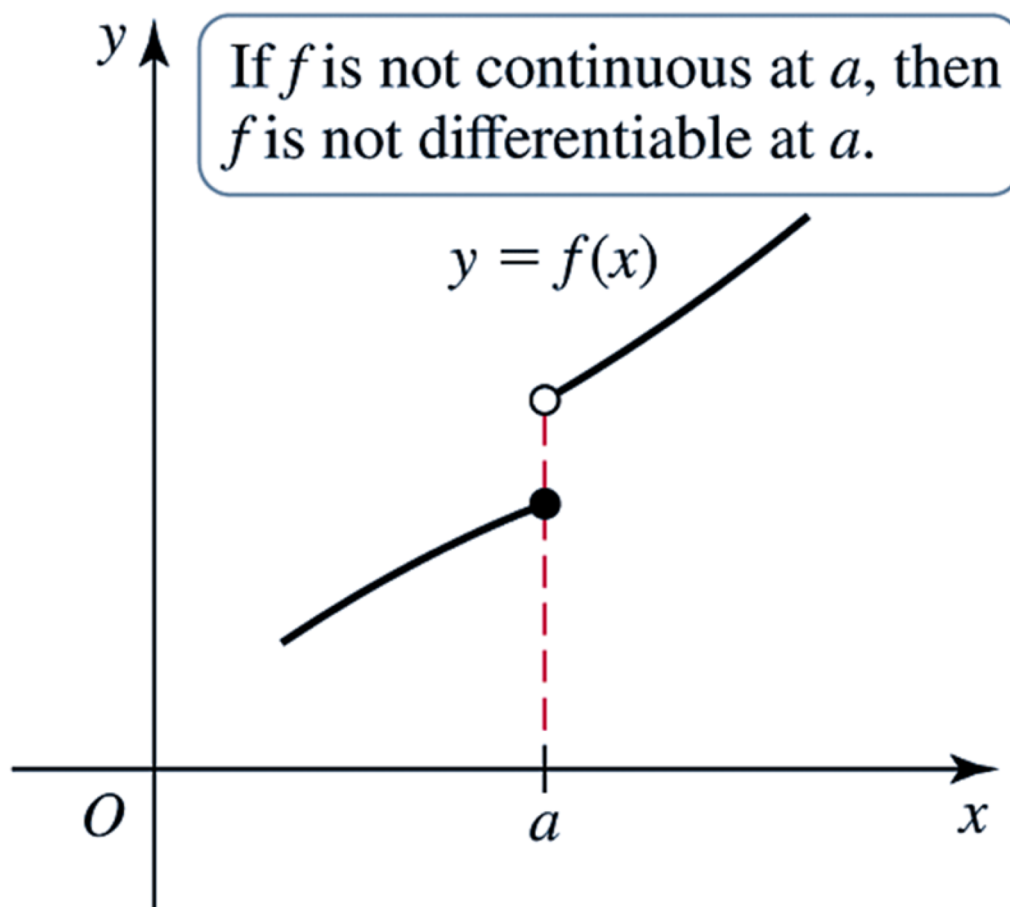


Theorem 3.1 Differentiable Implies Continuous



$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \rightarrow a} (x - a) + \lim_{x \rightarrow a} f(a)$$

Figure 3.24



Theorem 3.1 (Alternative Version) Not Continuous Implies Not Differentiable

If f is not continuous at a , then f is not differentiable at a .

$$A \Rightarrow B \quad \Leftarrow \Rightarrow \text{not } B \Rightarrow \text{not } A$$

Figure 3.25

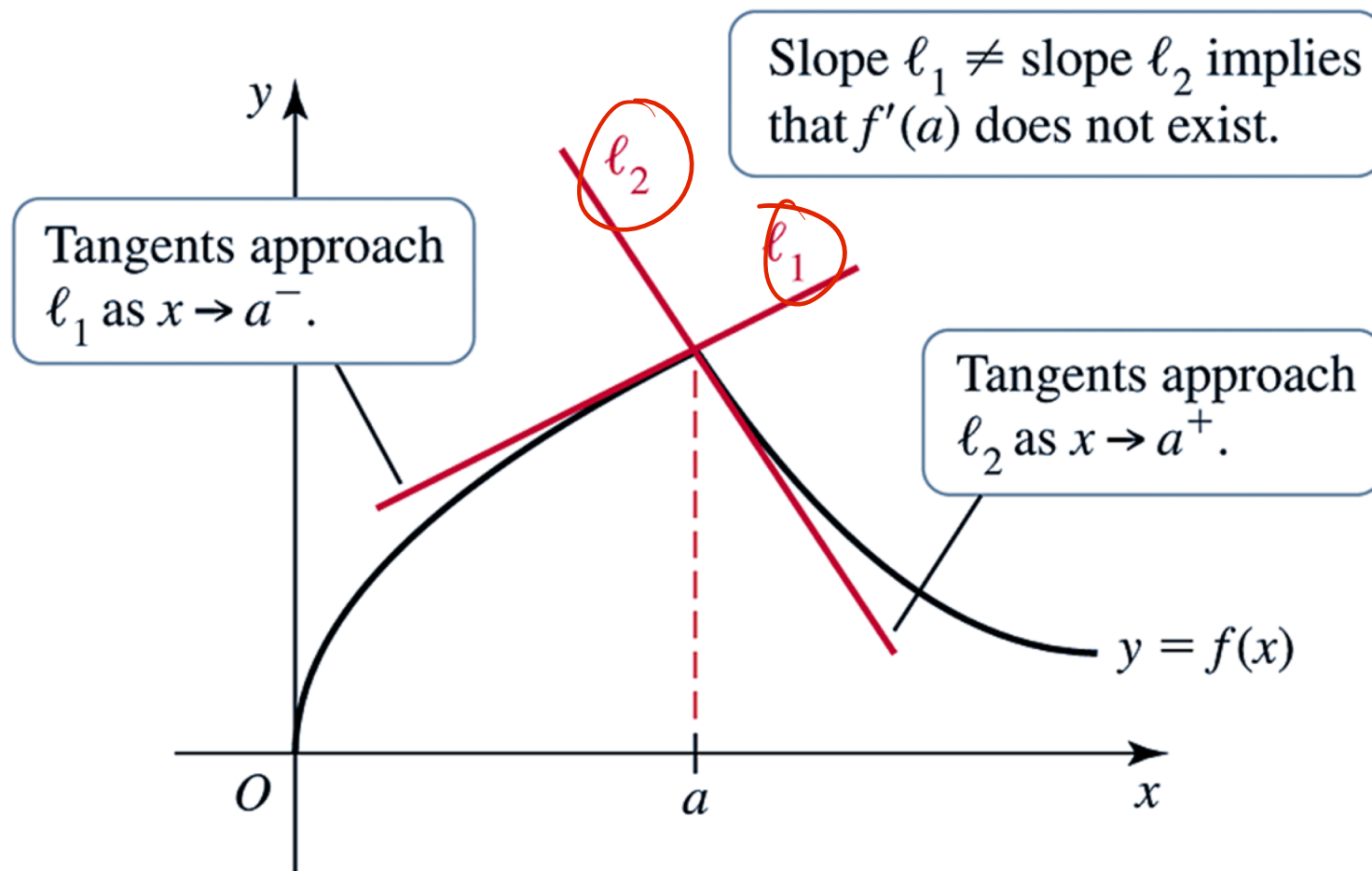
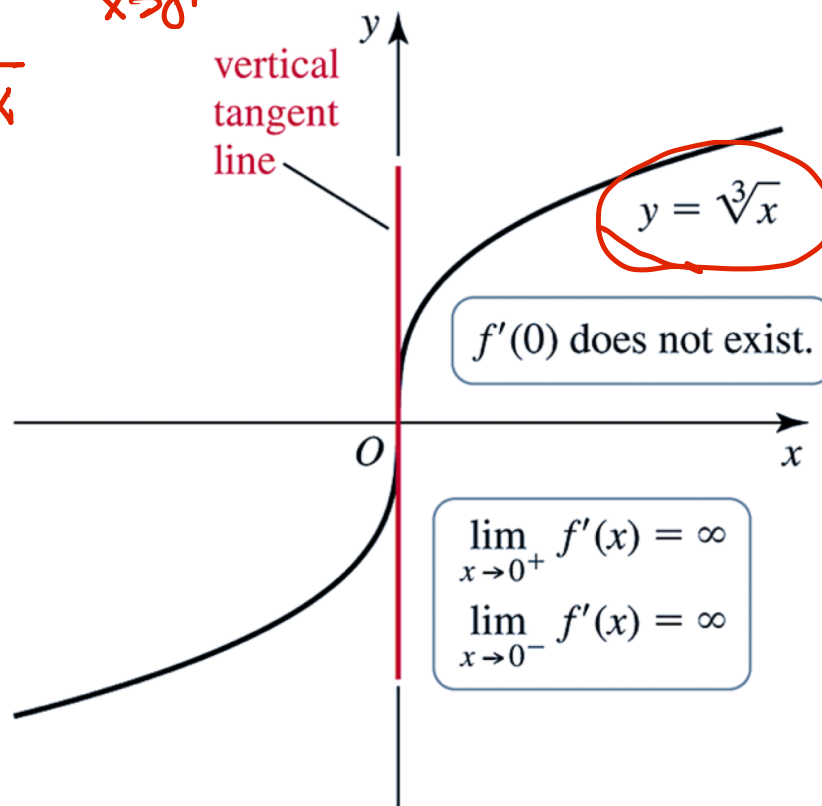
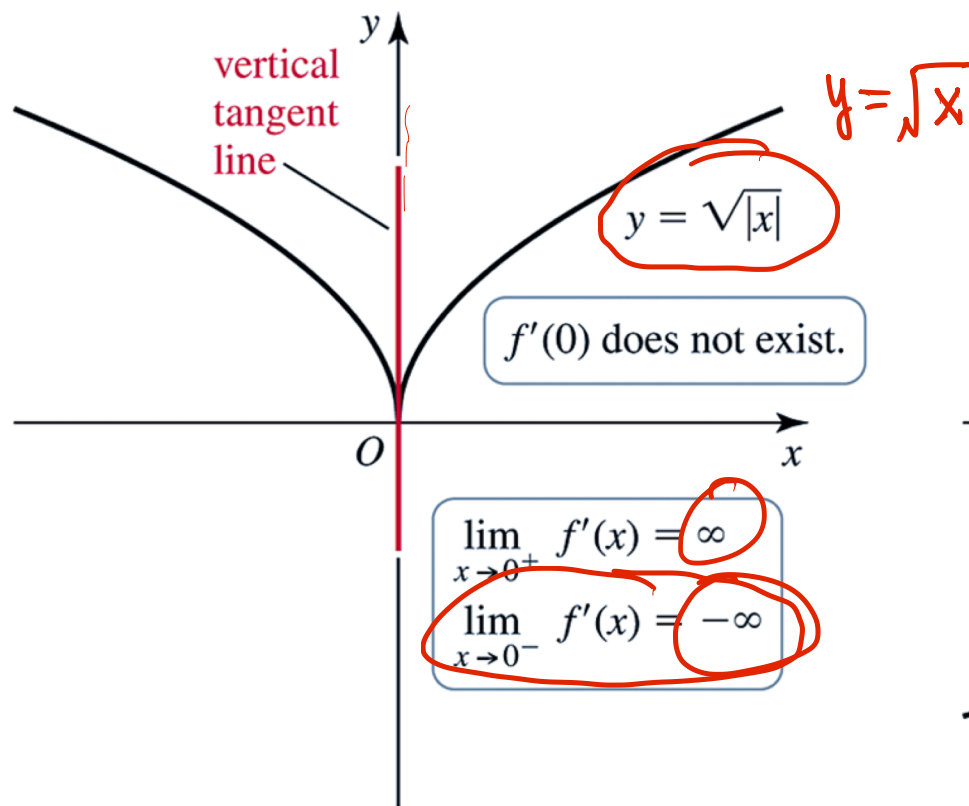


Figure 3.26

$$y = \sqrt{|x|} = \begin{cases} \sqrt{-x}, & x \leq 0 \\ \sqrt{x}, & x > 0 \end{cases}$$

$$(\sqrt{x})' = \frac{1}{2\sqrt{x}} \rightarrow \infty \quad x \rightarrow 0^+$$

$$\lim_{x \rightarrow 0^+} (\sqrt{x})' = +\infty$$



When Is a Function Not Differentiable at a Point?

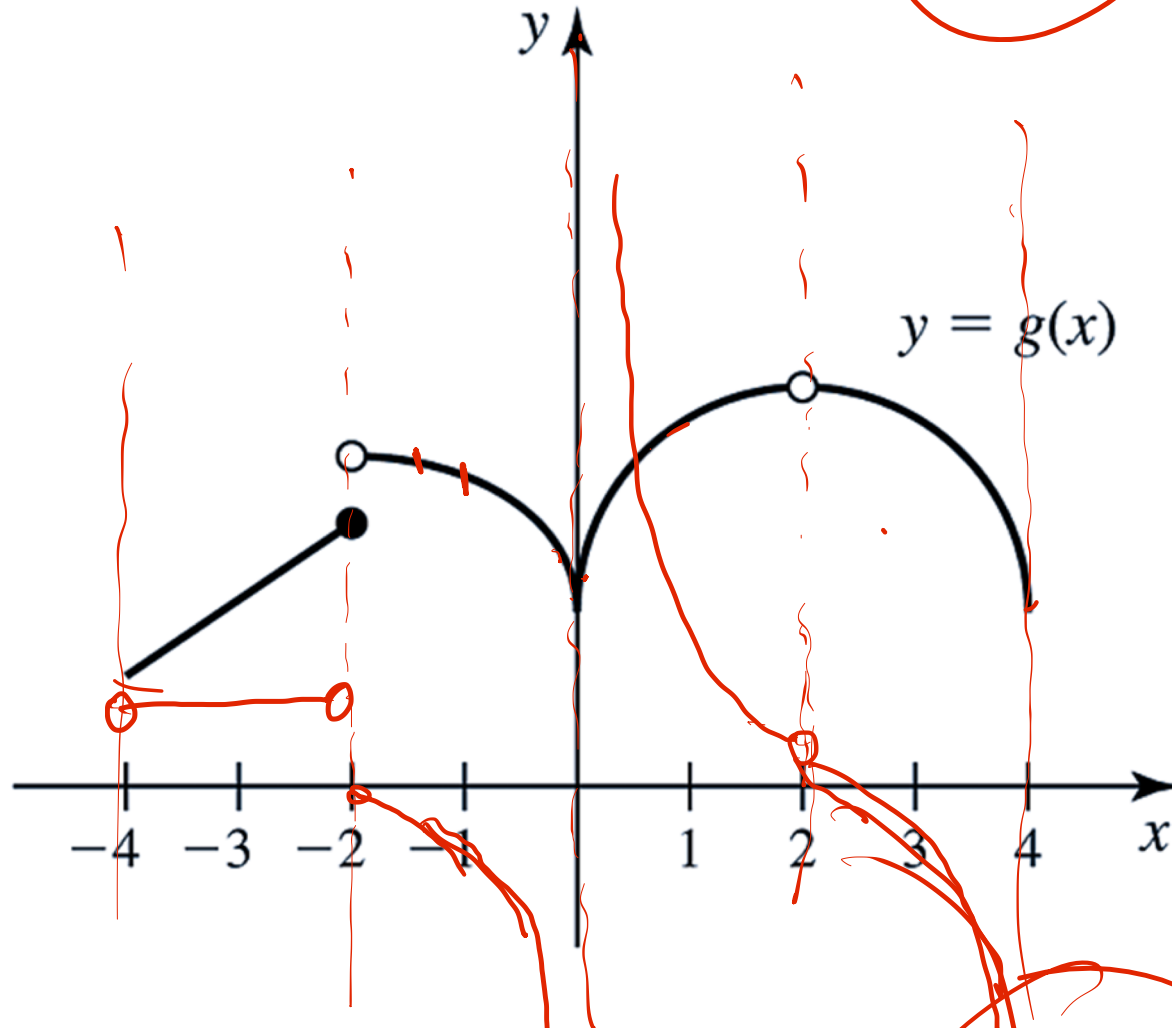
A function f is not differentiable at a if at least one of the following conditions holds:

- a. f is not continuous at a (Figure 3.24).
- b. f has a corner at a (Figure 3.25).
- c. f has a vertical tangent at a (Figure 3.26).

- Find $x \in (-4, 4)$ s.t. g is not cont.
- Find $x \in (-4, 4)$ s.t. g is not diff.
- sketch $g'(x)$

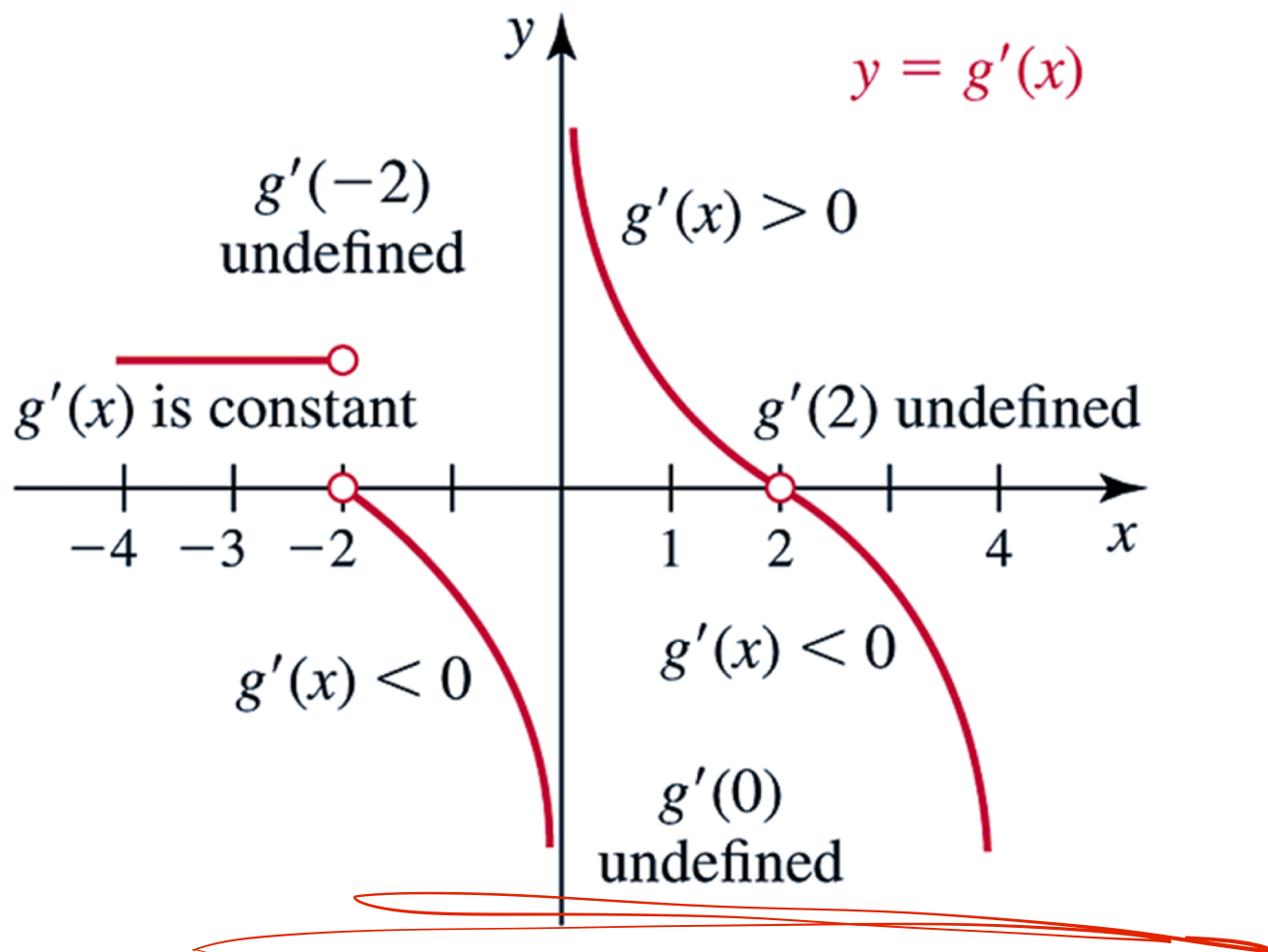
→ at $x = -2, 2$

Figure 3.27



$x \neq 0$

Figure 3.28



Section 3.3 Rules of Differentiation

(HW# 11, 14, 28, 29, 33, 35, 44, 47, 49, 51, 57, 67, 71, 72, 79, 82)

§3.4 The Product and Quotient Rules

(HW# 1, 2, 10, 11, 16, 20, 21, 22, 24, 29, 37, 56, 65, 73, 74, 92, 95, 97)

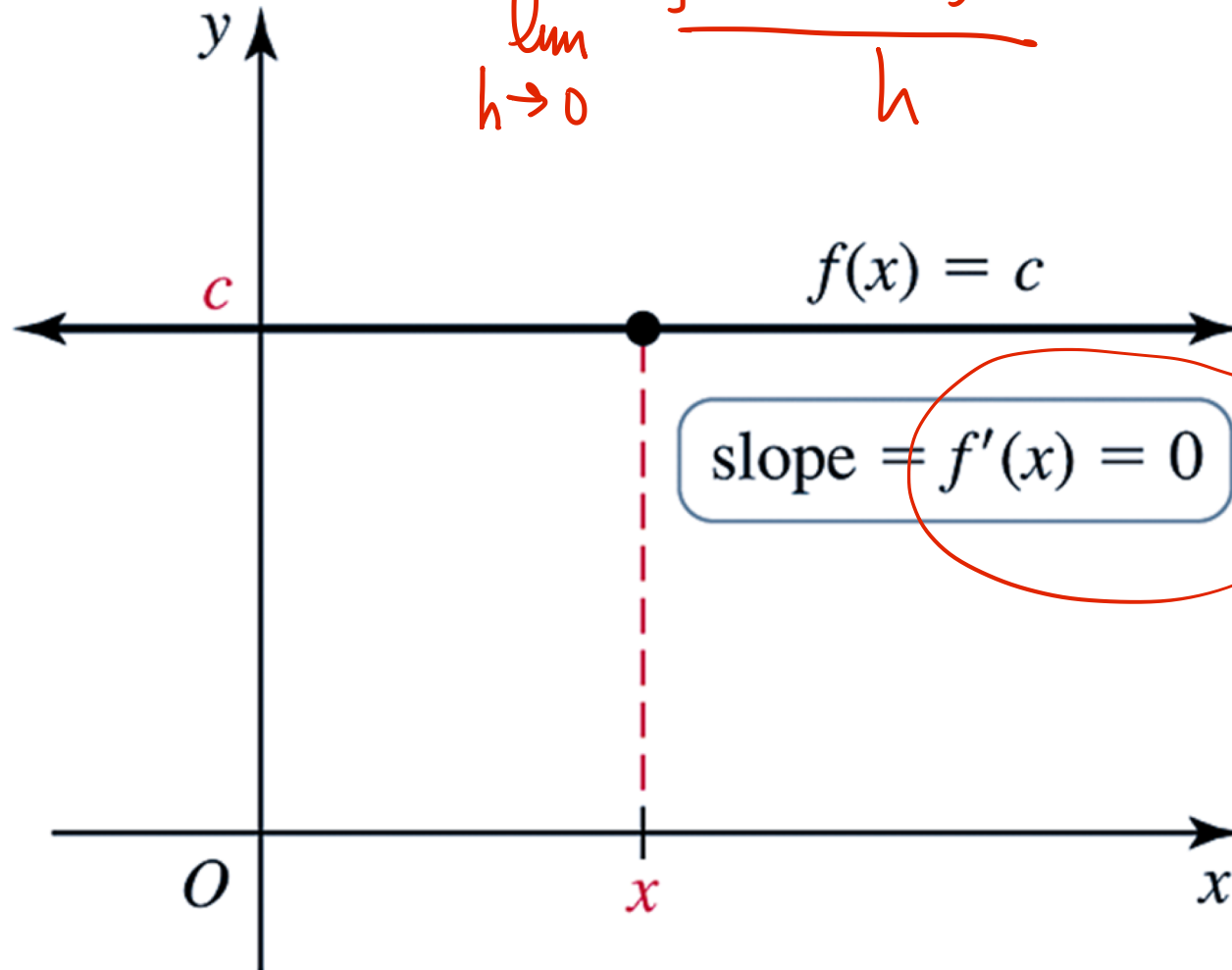
- • Linearity
- Rules of Power, Product, and Quotient

Theorem 3.2 (Constant Rule)

$$\frac{d}{dx}(c) = 0.$$

Figure 3.29

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \frac{c - c}{h} = 0$$



$$\frac{d}{dx}(x^2) = \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(2x+h)(\cancel{h})}{\cancel{h}} = 2x$$

Theorem 3.3 Power Rule

If n is a nonnegative integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

any real number

example $\frac{d}{dx}(x^5) = 5x^4$

$$\frac{d}{dx}(x^{\frac{1}{3}}) = \frac{1}{3}x^{-\frac{2}{3}}$$

$$(x+h)^3 - x^3 = (x+h-x)(\dots)$$

$$\frac{1}{3} \quad \frac{1}{3}$$

$$\bullet \frac{d}{dx}(cf(x)) = c \frac{d}{dx}f(x) = \lim_{h \rightarrow 0} \frac{cf(x+h) - cf(x)}{h} = c \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\bullet \frac{d}{dx}(f(x) + g(x)) = \lim_{h \rightarrow 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} = f' + g'$$

example $\frac{d}{dx}(3x^3 - 5x + 12x^{\frac{1}{2}}) = (3x^3)' - (5x)' + (12x^{\frac{1}{2}})' = 3 \cdot 3x^{3-1} - 5 \cdot 1 + 12 \cdot \frac{1}{2}x^{\frac{1}{2}-1} = 9x^2 - 5 + 6x^{-\frac{1}{2}}$

- Derivative of Exponential Function $f(x) = b^x$ at $x=0$

Table 3.1

$$f'(0) = \lim_{h \rightarrow 0} \frac{b^h - b^0}{h} = \lim_{h \rightarrow 0} \frac{b^h - 1}{h}$$

$$(b^x)' \Big|_{x=0} = ?$$

$\Downarrow$

$$(b^x)' = ?$$

$h \rightarrow 0$	$\frac{2^h - 1}{h}$	$\frac{3^h - 1}{h}$
1.0	1.000000	2.000000
0.1	0.717735	1.161232
0.01	0.695555	1.104669
0.001	0.693387	1.099216
0.0001	0.693171	1.098673
0.00001	0.693150	1.098618

$$(2^x)' \Big|_{x=0} \approx 0.693150$$

$$\approx (3^x)' \Big|_{x=0}$$

Figure 3.30

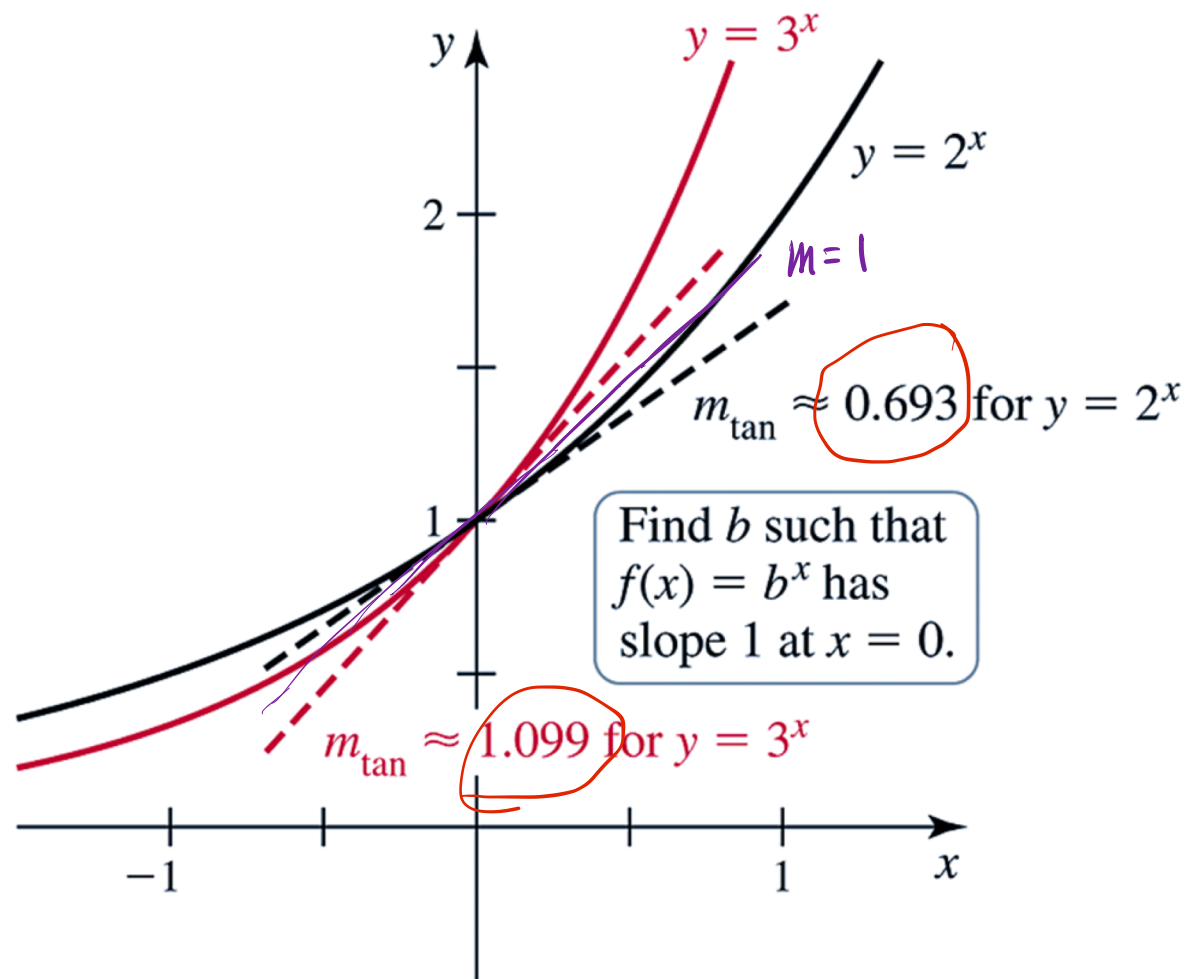
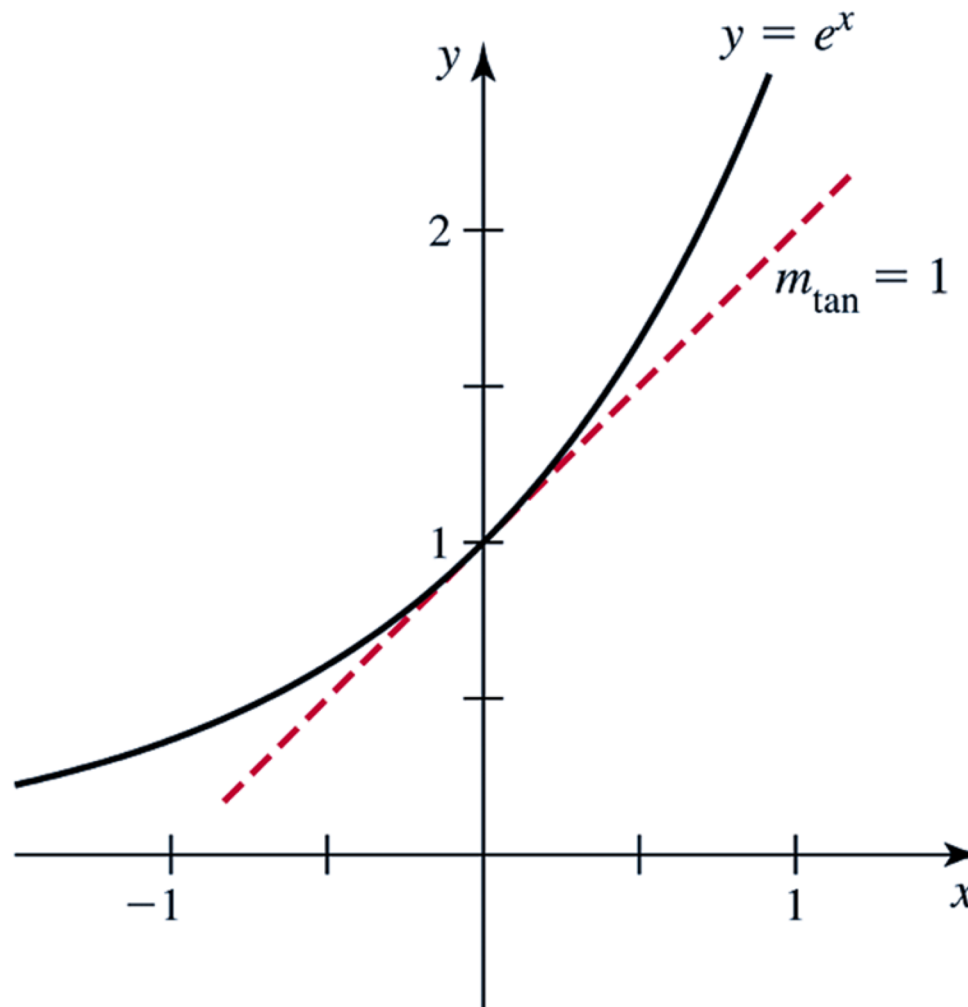


Figure 3.31



Definition The Number e

The number $e = 2.718281828459\dots$ satisfies

$$\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1.$$

It is the base of the natural exponential function

$$f(x) = e^x.$$

Theorem 3.6 The Derivative of e to the x Power

The function $f(x) = e^x$ is differentiable for all real numbers x , and

$$e^{x+h} = \underline{e^x} \cdot e^h$$

$$\frac{d}{dx}(e^x) = e^x.$$

$$\frac{d}{dx}(e^x) = \lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \rightarrow 0} \left(\underline{e^x} \right) \frac{(e^h - 1)}{h}$$

$$= e^x \lim_{h \rightarrow 0} \frac{e^h - 1}{h} = e^x$$

$$\begin{aligned} (3x^2 + e^x)' &= 3(x^2)' + (e^x)' \\ &= 6x + e^x \end{aligned}$$

Definition Higher-Order Derivatives

Assuming $y = f(x)$ can be differentiated as often as necessary, the **second derivative** of f is

$$\underline{f''(x)} = \frac{d}{dx} \left(\underline{f'(x)} \right).$$

For integers $n \geq 1$, the **n th derivative** of f is

$$f^{(n)}(x) = \frac{d}{dx} \left(f^{(n-1)}(x) \right).$$

example $f(x) = 2x + e^x$, $f'(x) = 2 + e^x$, $f''(x) = e^x$, $f'''(x) = e^x$

Theorem 3.7 Product Rule

If f and g are differentiable at x , then

$$\text{LHS} = \frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$

$$\text{LHS} = \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} = \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{[f(x+h) - f(x)]g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f(x)g(x+h) - f(x)g(x)}{h} + \lim_{h \rightarrow 0} \frac{[f(x+h) - f(x)]g(x+h)}{h}$$

example $(x^2 e^x)' = (x^2)'e^x + x^2(e^x)' = 2xe^x + x^2e^x$

Theorem 3.8 Quotient Rule

If f and g are differentiable at x and $g(x) \neq 0$,

then the derivative of $\frac{f}{g}$ at x exists and

$$\frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.$$

$$(g^{-1})' = (-1)(g^{-1-1}) \cdot g' = -\frac{g'}{g^2}$$

$$(f g^{-1})' = f' g^{-1} + f (g^{-1})' = f' g^{-1} - f g g^{-2}$$

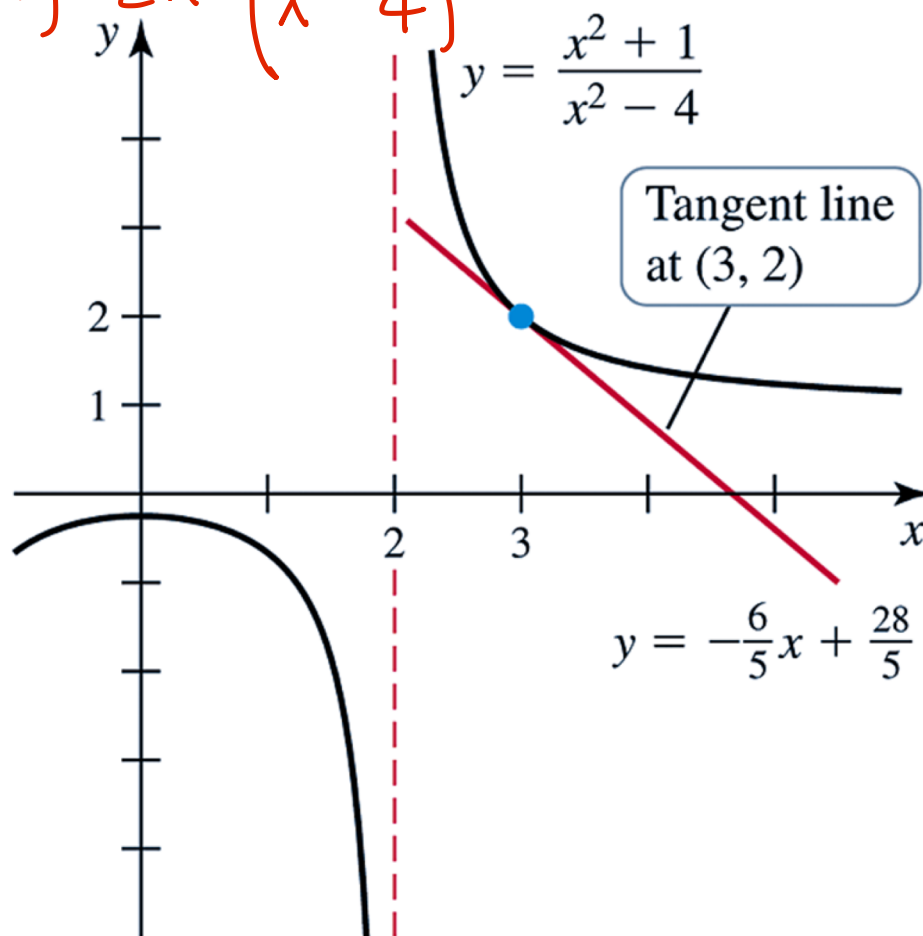
$$\left(\frac{1}{g}\right)' = ? \quad \left(1 = \frac{g}{g} = g \cdot \frac{1}{g}\right)' \quad 0 = g' \frac{1}{g} + g \left(\frac{1}{g}\right)'$$

$$\left(\frac{x^2 + 1}{x^2 - 4} \right)' = 2x \cdot (x^2 - 4)^{-1}$$

$$\left(\frac{1}{g} \right)' = -\frac{g'}{g^2} = -g'g^{-2}$$

Figure 3.34

$$= 2x(x^2 - 4)^{-1} - (x^2 + 1) \cdot 2x(x^2 - 4)^{-2}$$



$$(fg)' = f'g + fg', \quad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Section 3.5 Derivatives of Trigonometric Functions

(HW # 8, 12, 14, 19, 21, 23, 25, 29, 31, 40, 43, 48, 50, 54, 55, 59, 65, 66) AR 9.34, 9.49, 2.5.1)

- Basic limit

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

- derivatives of $\sin x$, $\cos x$, $\tan x$, ...

Theorem 3.10 Trigonometric Limits

$$\lim_{z \rightarrow 0} \frac{\sin z}{z} = \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = 0$$

$$\lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = \lim_{x \rightarrow 0} \frac{(\cos x + 1) \cdot 0}{(\cos x + 1) \cdot 0} = \lim_{x \rightarrow 0} \frac{0}{0}$$

$$\lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = \lim_{x \rightarrow 0} \frac{-\sin x}{x(\cos x + 1)} = -\lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \cdot \frac{1}{\cos x + 1} \right)$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \rightarrow 0} \frac{1}{\cos x + 1} = \frac{1}{2}$$

$$\lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = -\frac{1}{2}$$

$$\sin x \approx x$$

examples

$$\lim_{x \rightarrow 0} \frac{\sin 4x}{x} = \lim_{x \rightarrow 0} \frac{\sin 4x}{4x} \cdot \frac{1}{\frac{1}{4}} = \lim_{4x \rightarrow 0} \frac{\sin 4x}{4x} \cdot 4 = 4$$

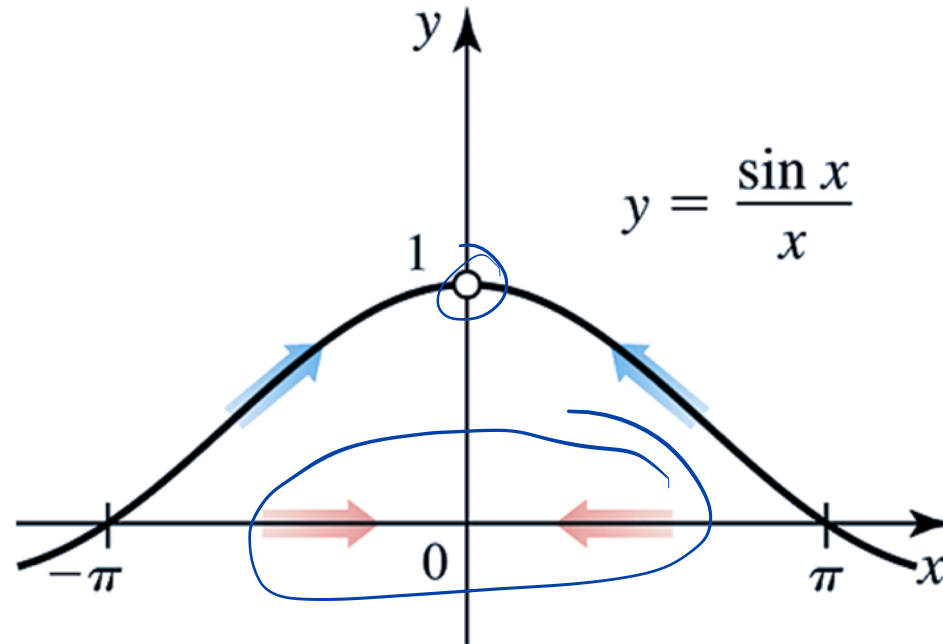
$$\lim_{x \rightarrow 0} \frac{\sin 3x}{\sin 5x} = \lim_{x \rightarrow 0} \frac{\sin 3x}{3x} \cdot \frac{1}{\frac{5x}{\sin 5x}} = \frac{3}{5}$$

$$= \lim_{x \rightarrow 0} \frac{3x}{5x} = \frac{3}{5}$$

Table 3.2

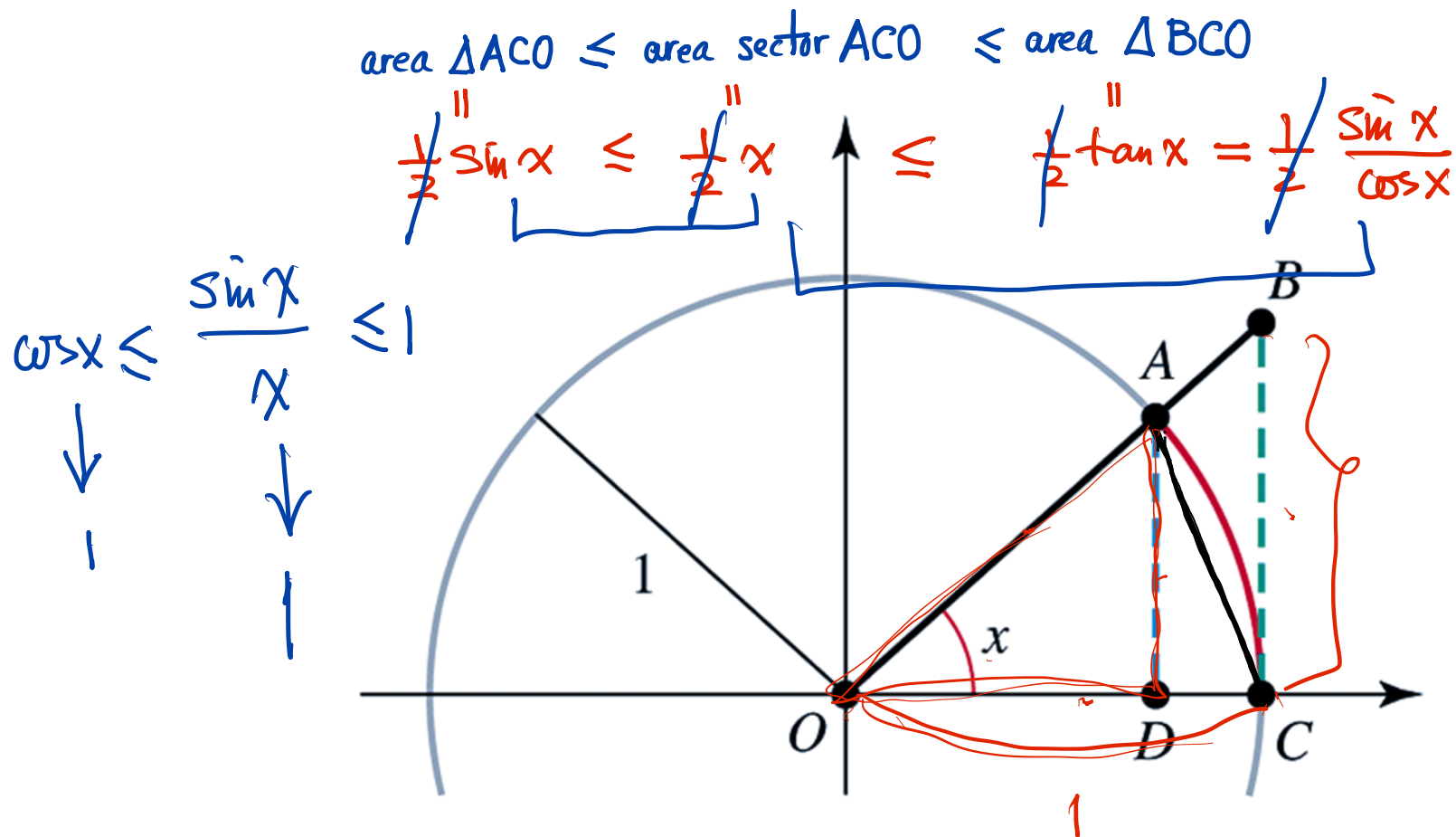
x	$\sin x$
± 0.1	0.9983341665
± 0.01	0.9999833334
± 0.001	0.9999998333

Figure 3.35



The graph of $y = \frac{\sin x}{x}$
suggests that $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$

Figure 3.36



• trigonometric limits

#15 $\lim_{x \rightarrow 0} \frac{\tan 5x}{x} = \lim_{x \rightarrow 0} \frac{\sin 5x}{(\cos 5x) 5x} = \frac{1 \cdot 5}{1} = 5$

Annotations: Red arrows point from $\tan 5x$ to $\sin 5x$ and from x to $5x$. Blue circles highlight $\sin 5x$ and $5x$. Blue arrows point from $\cos 5x$ to 1 and from the final denominator 1 to the final result 5.

#22 $\lim_{x \rightarrow 0} \frac{\sin ax}{bx} = \lim_{x \rightarrow 0} \frac{\sin ax \cdot a}{ax \cdot b} = \frac{a}{b}$

Annotations: Red circle around $\sin ax$ with an arrow pointing to 1. Red box around $a \neq 0$.

($b \neq 0$, a b - constants)

$a = 0$

$\lim_{x \rightarrow 0} \frac{\sin ax}{bx} = 0$

Theorem 3.11 Derivatives of Sine and Cosine

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\frac{d}{dx}(\sin x) = \cos x, \quad \frac{d}{dx}(\cos x) = -\sin x$$

$$= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h}$$

$$\begin{aligned} & \sin x \cos h + \sin h \cos x - \sin x \\ & \left(\frac{\sin h}{h} \right) \cos x + \sin x \left(\frac{\cos h - 1}{h} \right) \\ & \rightarrow \cos x \end{aligned}$$

$$= \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h}$$

$$\begin{aligned} & \cos x \cos h - \sin x \sin h - \cos x \\ & = \lim_{h \rightarrow 0} \left[-\sin x \frac{\sin h}{h} + \cos x \frac{\cos h - 1}{h} \right] \\ & = -\sin x + \cos x \cdot 0 = -\sin x \end{aligned}$$

Theorem 3.12 Derivatives of the Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\tan x) = \left(\frac{\sin x}{\cos x} \right)' = \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

#46 $\frac{d}{dx} \left(\frac{\tan x}{1 + \sec x} \right) = \frac{(\tan x)'(1 + \sec x) - \tan x (1 + \sec x)'}{(1 + \sec x)^2}$

$$= \frac{\sec^2 x (1 + \sec x) - \tan x \cdot \sec x \tan x}{(1 + \sec x)^2} = \frac{\sec^2 x + \sec^3 x - \tan^2 x \sec x}{(1 + \sec x)^2}$$

$$= \frac{\sec x (\sec x + 1)}{(1 + \sec x)^2} = \frac{\sec x}{1 + \sec x}$$

$\sec^2 x + \sec^3 x - \tan^2 x \sec x$
 $\sec^2 x + \sec x [\sec^2 x - \tan^2 x]$
 $\underline{\underline{1}}$

#67 $\lim_{x \rightarrow \frac{\pi}{4}} \frac{\tan x - 1}{x - \frac{\pi}{4}}$ (Hint: identify the limit as the derivative)

$$= \lim_{x \rightarrow \frac{\pi}{4}} \frac{\tan x - \tan \frac{\pi}{4}}{x - \frac{\pi}{4}} = (\tan x)' \Big|_{x = \frac{\pi}{4}}$$

$$= \sec^2 x \Big|_{x = \frac{\pi}{4}} = \frac{1}{\cos^2 \frac{\pi}{4}} = \frac{1}{\left(\frac{\sqrt{2}}{2} \right)^2} = 2$$

Section 3.6 Derivatives as Rates of Change + Part of §3.7

(HW §3.6 #1, 6, 18, 19, 25) (HW §3.7 #AR 37, 2, 10, 17, 20, 23, 25, 33, 35, 41, 44, 45, 59, 63, 75)
86, 89, 105, 108, 111 Part I of §3.7

Definition Velocity, Speed, and Acceleration

Suppose an object moves along a line with position $s = f(t)$. Then

the **velocity** at time t is $\underline{v} = \frac{ds}{dt} = f'(t)$,

the **speed** at time t is $\underline{v} = |v| = |f'(t)|$,

the **acceleration** at time t is $\underline{a} = \frac{dv}{dt} = \frac{d^2s}{dt^2} = f''(t)$.

Example 2 An object moves horizontally along a line: $s(t) = t^2 - 5t$, $t \in [0, 5]$.

a. determine when the object is stationary, moving to the left/right
b. graph $a(t)$. c. describe the motion of the object.

Figure 3.42

$$s(t) = t^2 - 5t, \quad t \in [0, 5]$$

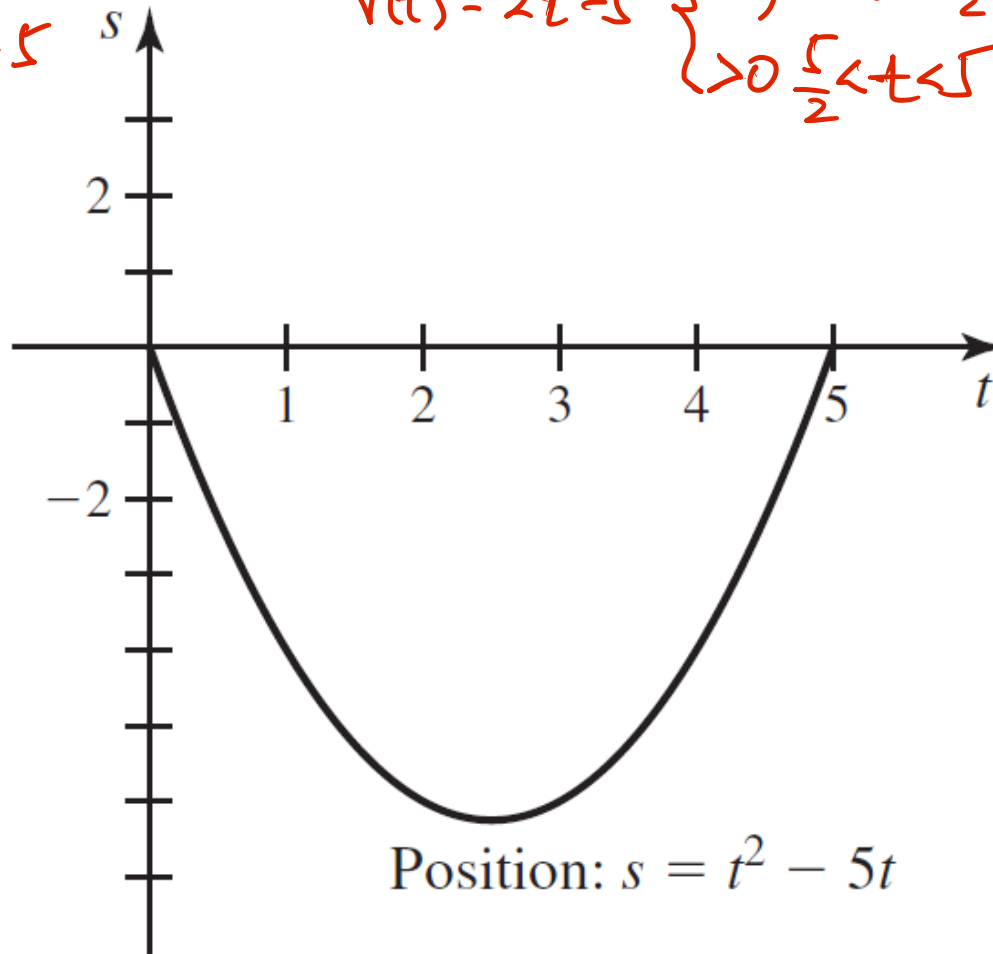
$$v(t) = s'(t) = 2t - 5$$

$$v = |2t - 5|$$

$$a(t) = v'(t) = 2$$

$$0 = v(t) = 2t - 5 \Rightarrow t = \frac{5}{2}$$

$$v(t) = 2t - 5 \begin{cases} < 0, & 0 < t < \frac{5}{2} \\ > 0, & \frac{5}{2} < t \leq 5 \end{cases}$$



$v(t)$ is increasing $[0, 5]$

Figure 3.43

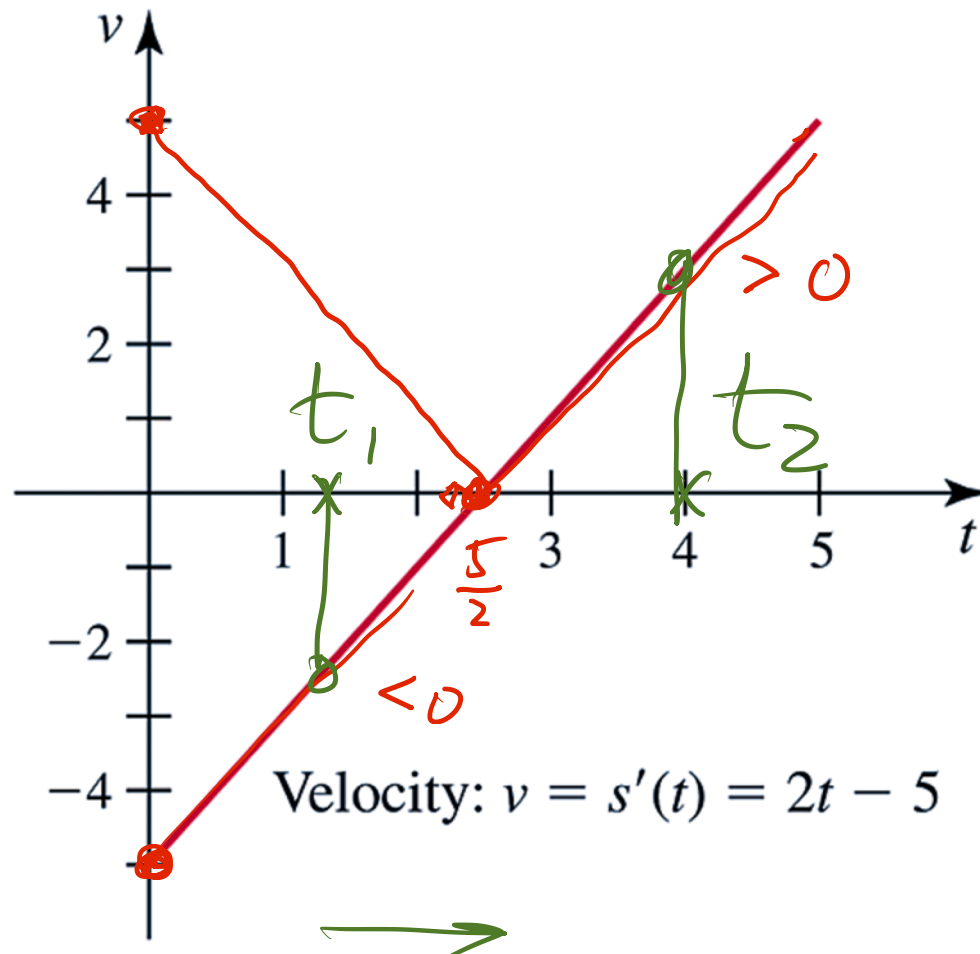
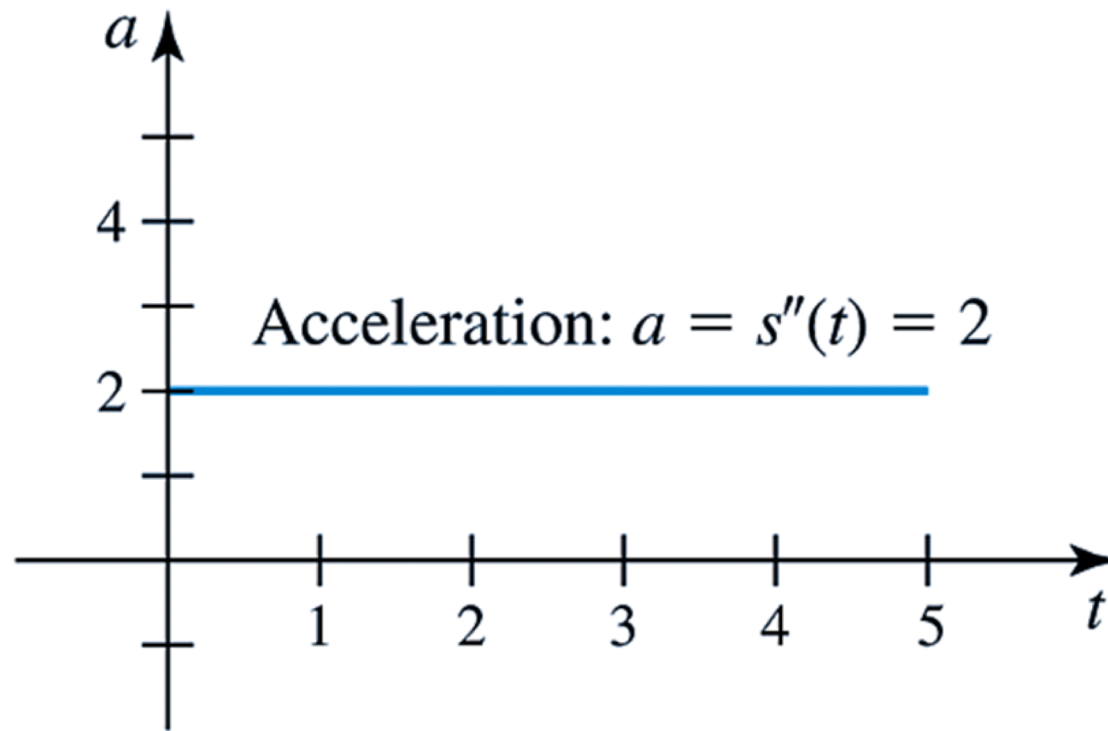


Figure 3.44

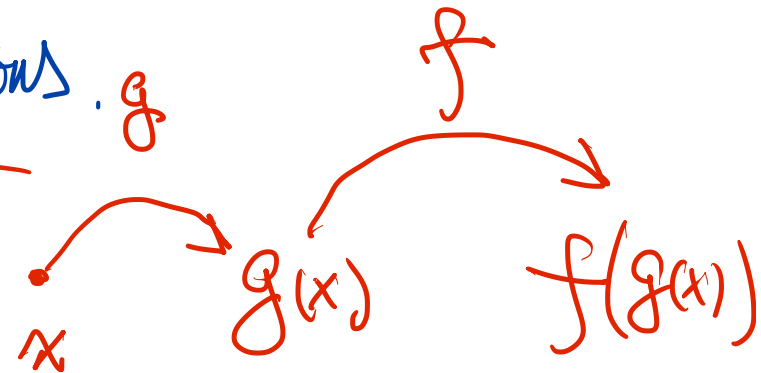


Section 3.7 The Chain Rule

(HW# 49, 53, 55, 58, 61, 66, 69, 71, 73, 97, 101, 103)

- derivative of composition functions.

$$(f \circ g)(x) = f(g(x))$$



Theorem 3.13 The Chain Rule

Suppose $y = f(u)$ is differentiable at $u = g(x)$ and $u = g(x)$ is differentiable at x . The composite function $y = f(g(x))$ is differentiable at x , and its derivative can be expressed in two equivalent ways.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad (1)$$

$$\frac{d}{dx} (f(g(x))) = f'(g(x)) \cdot g'(x) \quad (2)$$

$$= \lim_{h \rightarrow 0} \frac{f(g(x+h)) - f(g(x))}{h} = \lim_{h \rightarrow 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h}$$

$$= \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h}$$

$$= \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h}$$

Procedure Using the Chain Rule

Assume the differentiable function $y = f(g(x))$ is given.

1. Identify an outer function f and an inner function g , and let $u = g(x)$.

2. Replace $g(x)$ with u to express y in terms of u :

$$y = f(\underbrace{g(x)}_u) \Rightarrow y = f(u).$$

3. Calculate the product $\frac{dy}{du} \cdot \frac{du}{dx}$.

4. Replace u with $g(x)$ in $\frac{dy}{du}$ to obtain $\frac{dy}{dx}$.

Example 1 Compute $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ $u = ?$

a. $y = (5x+4)^{\frac{3}{2}}$
 $= u^{\frac{3}{2}}$

$u = g(x) = 5x+4$

$y'(x) = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{3}{2} u^{\frac{3}{2}-1} \cdot 5 = \frac{15}{2} (5x+4)^{\frac{1}{2}}$

b. $y = \sin^3 x = (\sin x)^3$
 $y'(x) = 3 (\sin x)^2 \cdot \cos x$

c. $y = \sin x^3 = \sin(x^3)$
 $y' = \cos(x^3) \cdot 3x^2$

Example 2 Compute $\frac{d}{dx} f(g(x)) = f'(g(x)) g'(x)$

a. $(6x^3 + 3x + 1)^{10} = y(x)$

$$y'(x) = 10 \left(\right)^{10-1} \cdot (6 \cdot 3x^2 + 3)$$

b. $\sqrt{5x^2 + 1} = (5x^2 + 1)^{\frac{1}{2}} = y(x)$

$$y'(x) = \frac{1}{2} (5x^2 + 1)^{\frac{1}{2} - 1} \cdot (5 \cdot 2x)$$

c. $\left(\frac{5t^2}{3t^2 + 2} \right)^3 = y$

$$y' = 3 \left(\frac{5t^2}{3t^2 + 2} \right)^2 \cdot \frac{(5 \cdot 2t)(3t^2 + 2) - 5t^2(3 \cdot 2t)}{(3t^2 + 2)^2}$$

d. $e^{-3x} = e^{(-3x)} = y(x)$

$$y'(x) = e^{(-3x)} \cdot (-3x)' = -3e^{-3x}$$

$$\left[f(g(x)) \right]' = f'(g(x)) \cdot g'(x)$$

Example 3 Let $h(x) = f(g(x))$. Compute $h'(1)$ and $h'(2)$.

Table 3.3

$$h'(x) = f'(g(x)) g'(x)$$

$$h'(1) = f'(\underbrace{g(1)}) \underbrace{g'(1)} = \underbrace{f(2)} \cdot 3 = 7 \cdot 3 = 21$$

x	$f'(x)$	$g(x)$	$g'(x)$
<u>1</u>	5	<u>2</u>	<u>3</u>
<u>2</u>	7	1	4

Theorem 3.14 Chain Rule for Powers

If g is differentiable for all x in its domain and p is a real number, then

$$\frac{d}{dx} \left(\left(\underline{g(x)} \right)^{\underline{p}} \right) = \underline{p} \left(\underline{g(x)} \right)^{p-1} \underline{g'(x)}.$$

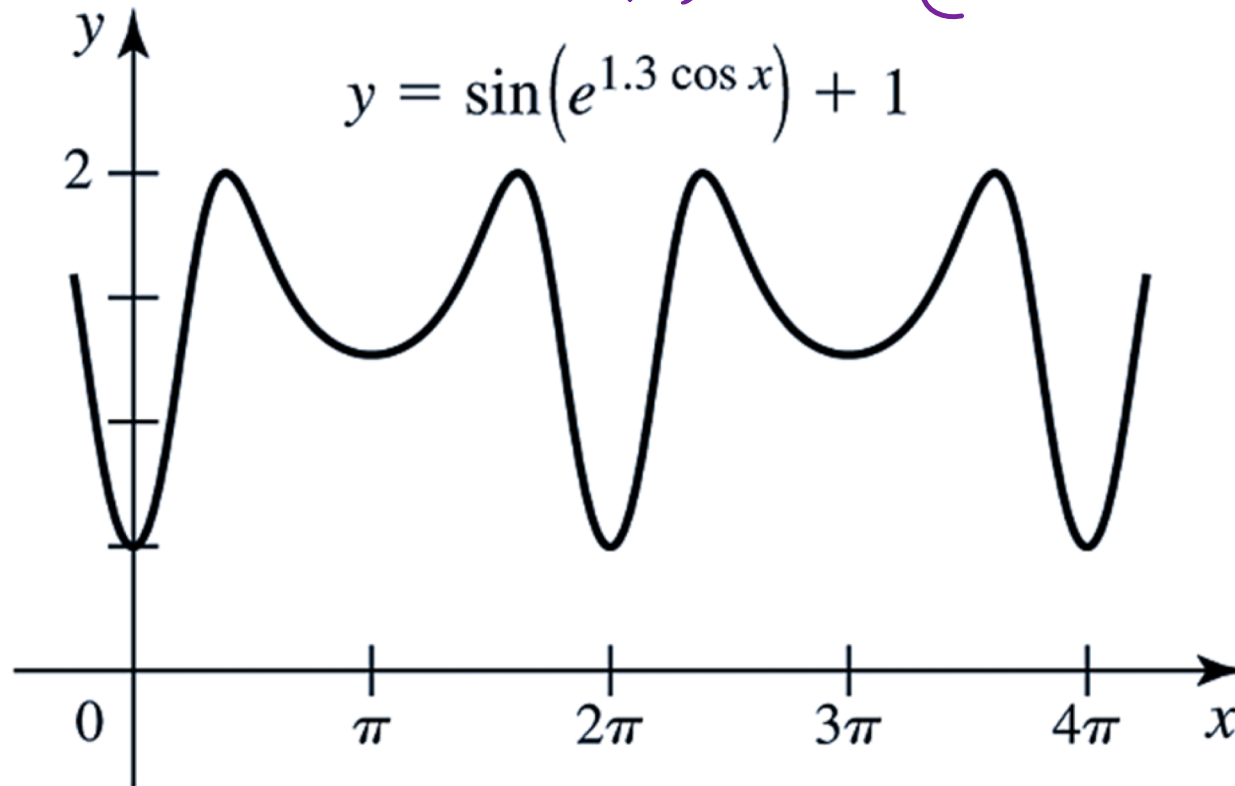
$$\frac{d}{dx} f \left(g(h(x)) \right) = f' \left(g(h(x)) \right) \cdot g'(h(x)) \cdot h'(x)$$

Example 6 $f(x) = \underline{\underline{\sin}} \left(\underline{\underline{e^{1.3 \cos x}}} \right) + \underline{\underline{1}}$

Figure 3.56

$$f'(x) = \cos \left[\left(e^{1.3 \cos x} \right)^{1.3} \cdot e^{1.3 \cos x} \cdot 1.3 (-\sin x) \right]$$

$$= -1.3 \sin x e^{1.3 \cos x} \cos \left[e^{1.3 \cos x} \right]$$



#60 $y(x) = \sqrt{x + \sqrt{x + \sqrt{x}}} = \left[x + \left(x + x^{\frac{1}{2}} \right)^{\frac{1}{2}} \right]^{\frac{1}{2}}$

$y'(x) = \frac{1}{2} \left[x + \left(x + x^{\frac{1}{2}} \right)^{\frac{1}{2}} \right]^{\frac{1}{2} - 1} \cdot \left[1 + \frac{1}{2} \left(x + x^{\frac{1}{2}} \right)^{-\frac{1}{2}} \cdot \left(1 + \frac{1}{2} x^{-\frac{1}{2}} \right) \right]$

#62 $y(x) = \left(\underline{f}(\underline{g}(\underline{x^m})) \right)^n$

$y'(x) = n \left(f(g(x^m)) \right)^{n-1} \cdot f'(g(x^m)) \cdot g'(x^m) \cdot m x^{m-1}$

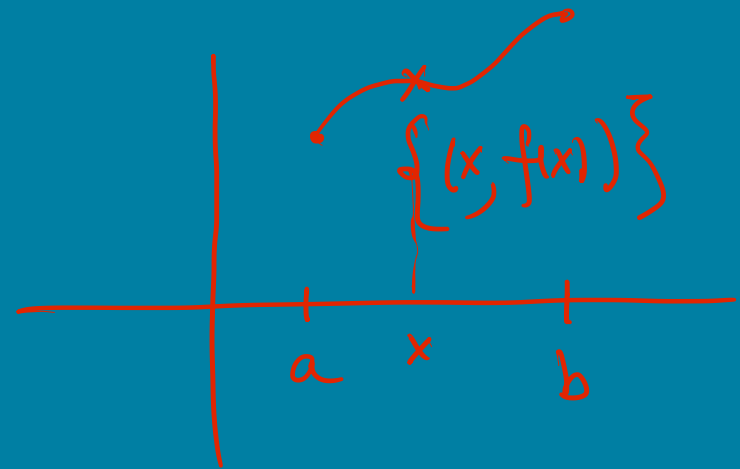
$\frac{d^2}{dx^2} \underline{f(g(x))} = \frac{d}{dx} \left[\underline{f'(g(x))} \cdot \underline{g'(x)} \right]$

$= \left[\underline{f'(g(x))} \right]' g'(x) + f'(g(x)) \cdot [g'(x)]'$

$= \underline{f''(g(x))} \cdot \underline{g'(x)} \cdot g'(x) + f'(g(x)) g''(x)$

$$y = f(x)$$

$$\frac{dy}{dx} = f'(x)$$



Section 3.8 Implicit Differentiation

(HW#13, 17, 18, 27, 31, 32, 39, 46, 50, 55, 65, 78, 82, 89)

- explicit differentiation
- implicit differentiation

Example 1 $x^2 + y^2 = 1$ compute $\frac{dy}{dx} = ?$

$$\frac{d}{dx} (x^2 + y^2) = 1$$

Figure 3.57 (a & b)

$$= \frac{dx^2}{dx} + \frac{dy^2}{dx} = 0$$

$$= 2x + 2(y(x))^{2-1} \cdot \frac{dy}{dx}$$

$$= 2x + 2y(x) \cdot y'(x) = 0$$

$$y'(x) = -\frac{2x}{2y(x)}$$

$$y'(x) = -\frac{x}{y(x)}$$

$x^2 + y^2 = 1$ fails the vertical line test.

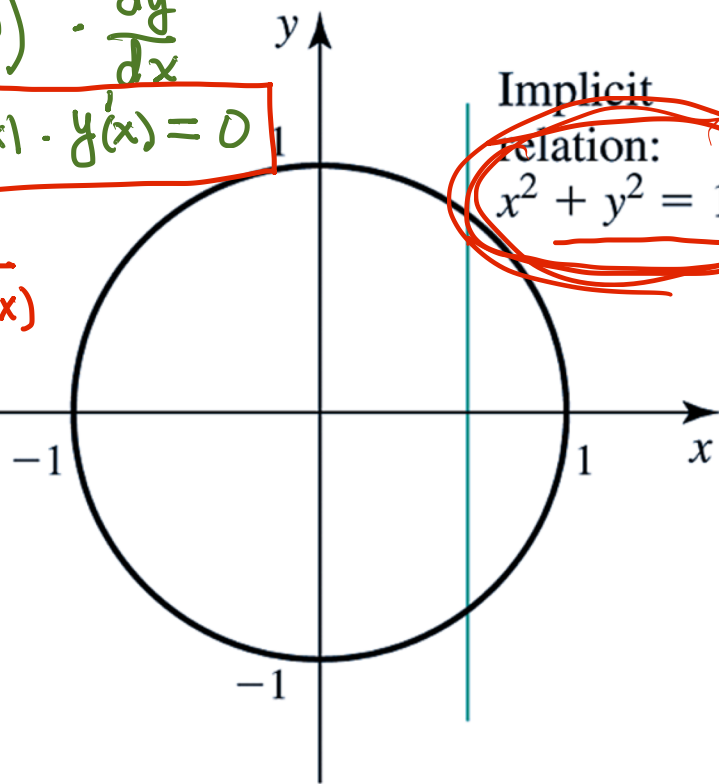
differentiate
solve linear eq

solve nonlinear eq

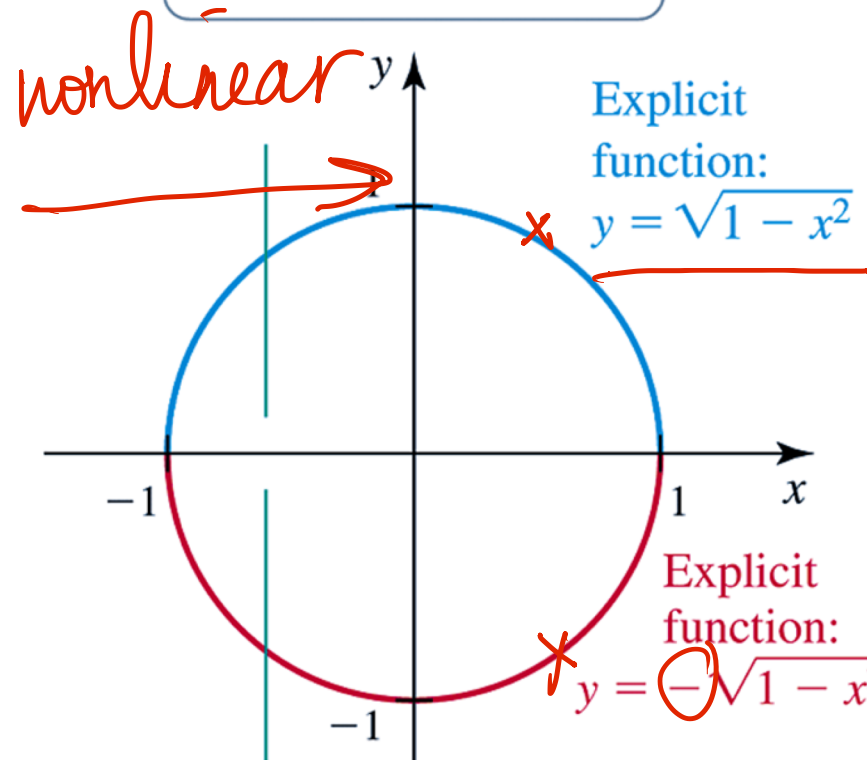
compute der

Each semicircle passes the vertical line test.

$$\frac{dy}{dx} = \frac{1}{2} (1-x^2)^{-\frac{1}{2}} \cdot (-2x)$$



Implicit relation:
 $x^2 + y^2 = 1$



nonlinear

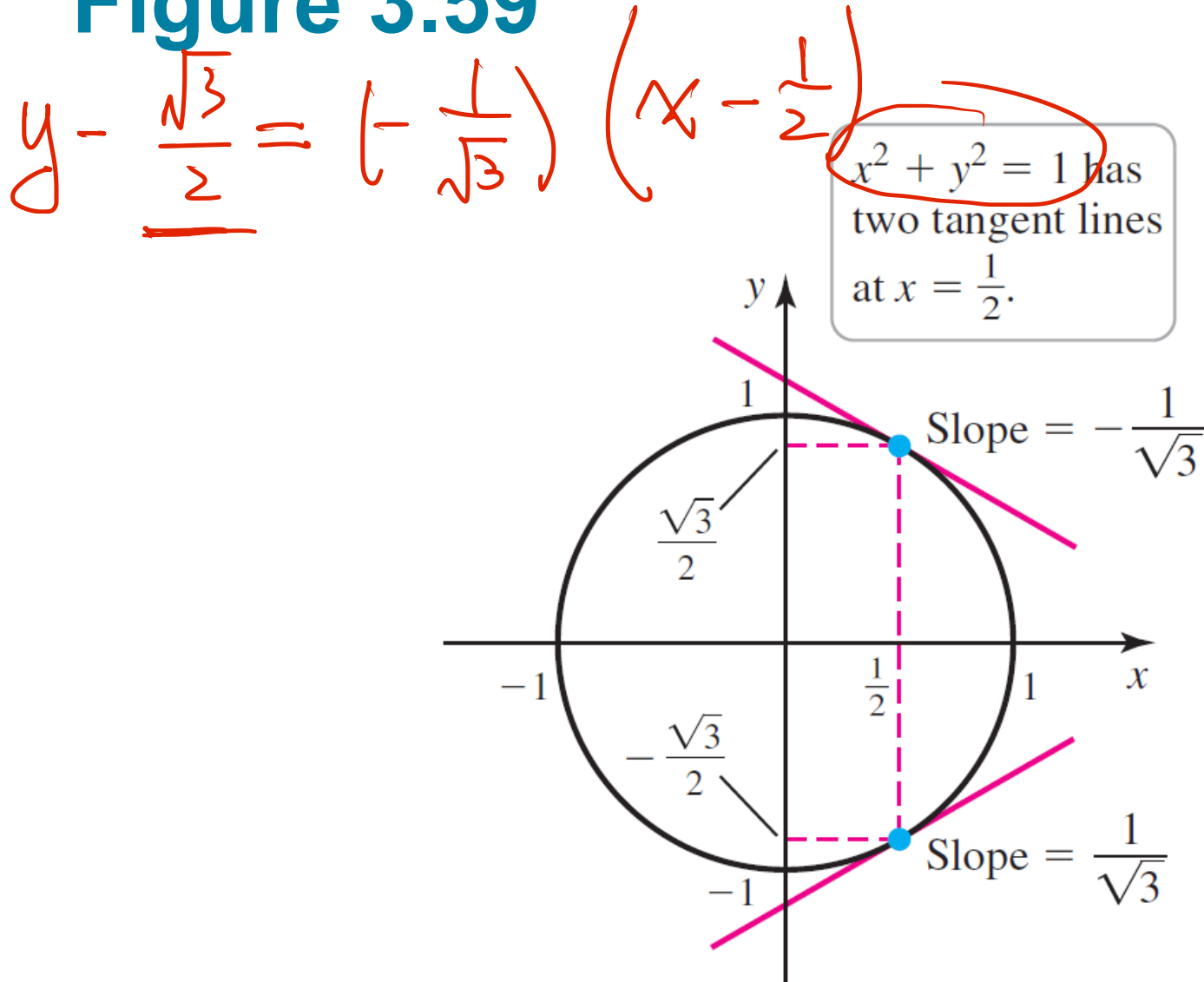
Explicit function:
 $y = \sqrt{1 - x^2}$

Explicit function:
 $y = -\sqrt{1 - x^2}$

$$\frac{dy}{dx} = -\frac{1}{2} (1-x^2)^{-\frac{1}{2}} \cdot (-2x)$$

Find the equation of the tangent line at $(\frac{1}{2}, \frac{\sqrt{3}}{2})$.

Figure 3.59



$$y' = -\frac{x}{y}$$

$$y'(\frac{1}{2}) = -\frac{\frac{1}{2}}{y(\frac{1}{2})}$$

$$= -\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$

$$= -\frac{1}{\sqrt{3}}$$

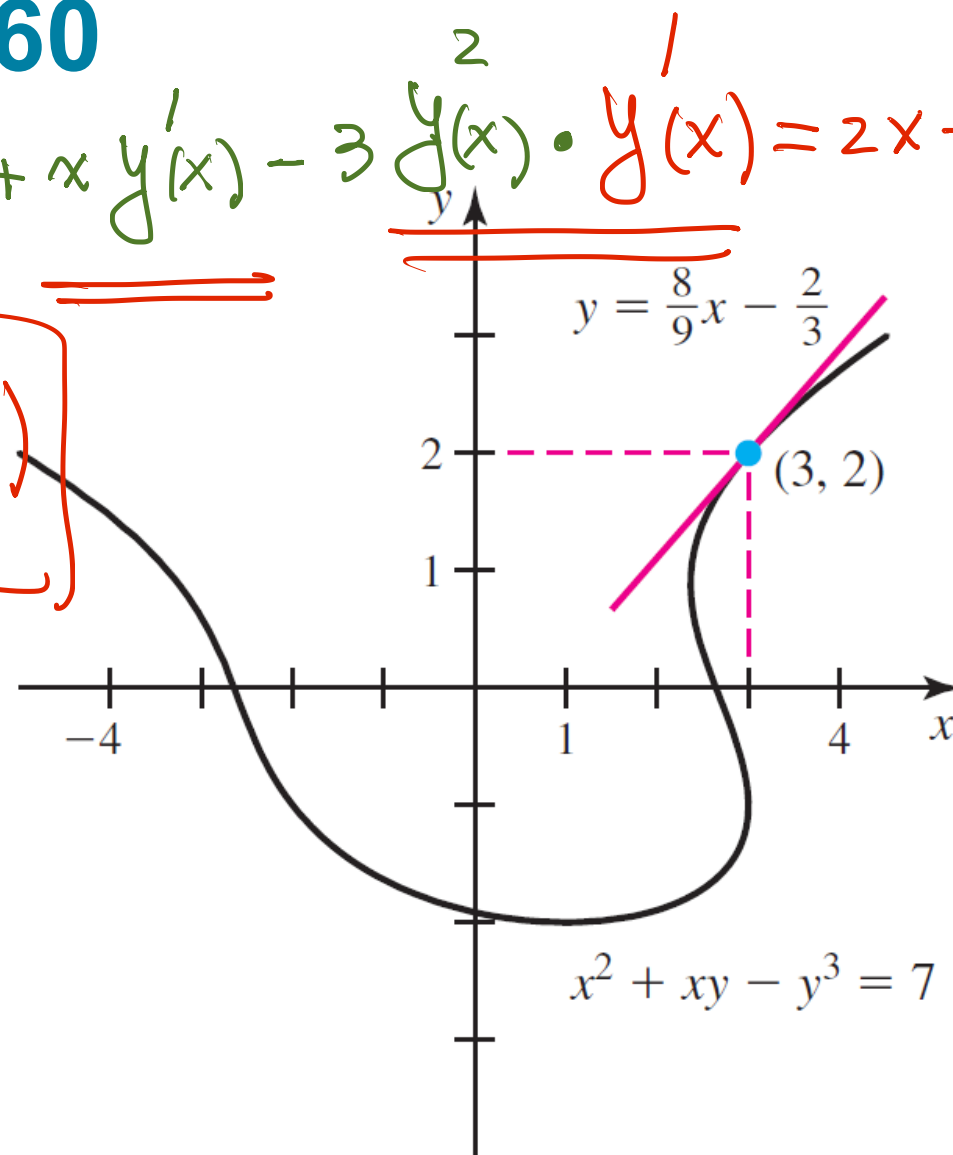
Example 3 $x^2 + xy - y^3 = 7$. (1) $y'(x)$ and (2) the tangent line at $(3, 2)$

$$\frac{d}{dx} (x^2 + xy - y^3) = 0$$

Figure 3.60

$$= 2x + y + xy'(x) - 3y^2 \cdot y'(x) = 2x + y + [x - 3y^2] y'(x)$$

$$y = 2 + \frac{8}{9}(x - 3)$$



$$y' = -\frac{2x + y}{x - 3y^2}$$

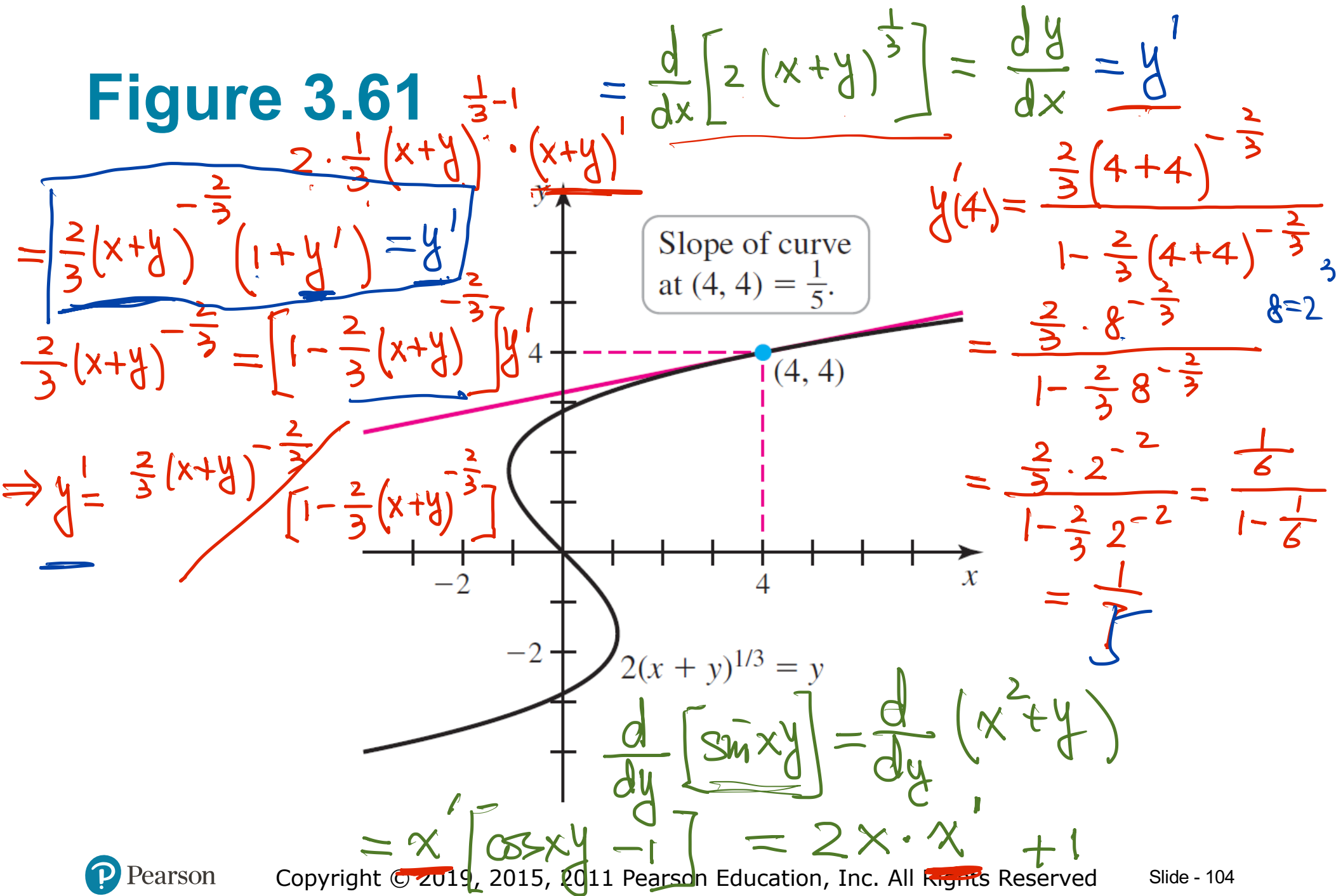
$$y'(3) = -\frac{2 \cdot 3 + 2}{3 - 3 \cdot 2^2}$$

$$= -\frac{8}{-9}$$

$$= \frac{8}{9}$$

Example 4 Find the slope of the curve $2(x+y)^{\frac{1}{3}} = y$ at (4, 4).

Figure 3.61



Example 2 Find $y'(x)$ when $\sin(xy) = x^2 + y$

$$\cos(xy) \cdot (xy)' = \cos(xy) \cdot (y + x y') = 2x + y'$$

$$y' [x \cos xy - 1] = 2x + y \cos xy \quad \left[\frac{dx}{dy} \right]$$

$$y' = \frac{2x + y \cos xy}{x \cos xy - 1}$$

Example 5 Find $\frac{d^2 y}{dx^2}$ if $x^2 + y^2 = 1$

$$\frac{d}{dx} (x^2 + y^2 = 1) \Rightarrow$$

$\Rightarrow$

$$2x + 2y y' = 0$$

$$\frac{d}{dx} (2x + 2y y' = 0)$$

$$\Rightarrow y' = -\frac{x}{y}$$

$$y'' = \frac{y - x y'}{y^2}$$

$$1 + y \cdot y' + y y'' = 0$$

$$\Rightarrow y'' = -\frac{1 + (y')^2}{y}$$

Section 3.9 Derivatives of Logarithmic and Exponential Functions

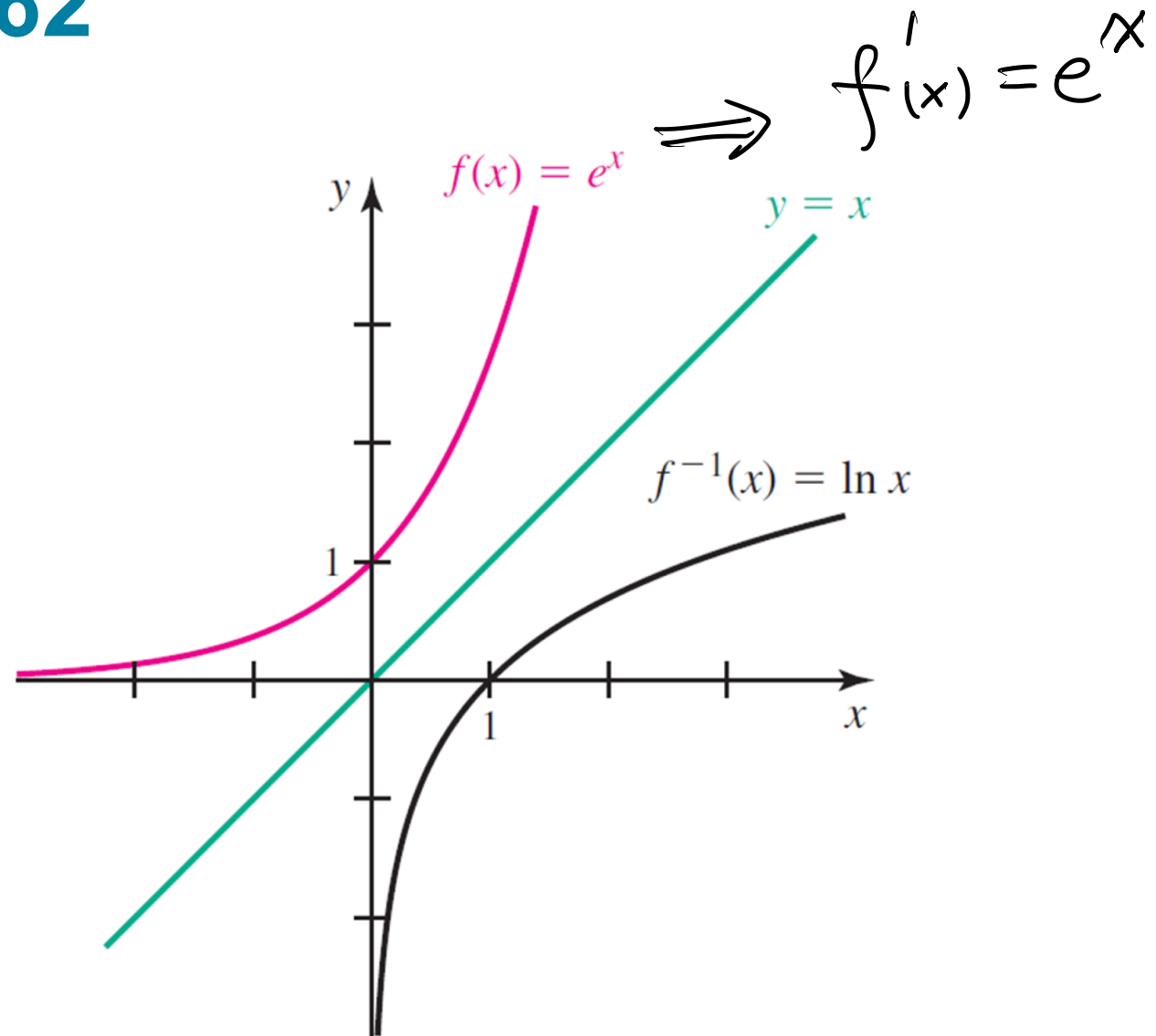
(HW Part I # AR8.40, 5, 16, 19, 23, 27, 29, 34, 36, 37, 47, 91, 105, 107)
Part II # 51, 53, 59, 75, 76, 79, 83, 85

- $\frac{d}{dx}(\log_b |x|) = \frac{1}{x \ln b}$
- $\frac{d}{dx}(b^x) = b^x \ln b$

$$b^x = e^{\ln b^x} = e^{x \ln b}$$

$$y = \ln x \iff e^y = e^{\ln x} = x$$

Figure 3.62



Inverse Properties for e to the x Power and Natural Log of x

1. $e^{\ln x} = x$, for $x > 0$, and $\ln(e^x) = x$, for all x .
2. $y = \ln x$ if and only if $x = e^y$.
3. For real numbers x and $b > 0$, $b^x = e^{\ln b^x} = e^{x \ln b}$.

- $y = \ln x$, compute $\frac{dy}{dx} = \frac{d}{dx}(\ln x) = \lim_{h \rightarrow 0} \frac{\ln(x+h) - \ln x}{h}$
 $\frac{d}{dx} \boxed{x = e^{y(x)}}$ $1 = e^y \cdot y' \Rightarrow y' = \frac{1}{e^y} = \frac{1}{x}$
 $\boxed{\frac{d}{dx} \ln x = \frac{1}{x}}$
- $y = \ln|x|$, compute $\boxed{\frac{d}{dx}(\ln|x|) = \frac{1}{x}}$
 $= \begin{cases} \ln x, & x > 0 \\ \ln(-x), & x < 0 \end{cases}$ $(\ln x)' = \frac{1}{x}$
 $(\ln(-x))' = \frac{1}{(-x)} \cdot (-x)' = \frac{1}{x}$
 $\boxed{x \neq 0}$
 $(\ln|x|)' = \frac{1}{|x|} \cdot (|x|)'$
- derive $(\ln x)' = \frac{1}{x}$ using $\ln x$ and e^x are inverse each other
 $y = \ln x \Rightarrow \underbrace{z(x)}_{z'(x)} = e^{y(x)} = e^{\ln x} = x$
 $z'(x) = e^{y(x)} \cdot y'(x) \Rightarrow y' = \frac{z'(x)}{e^{y(x)}} = \frac{1}{x}$

Theorem 3.15 Derivative of Natural Log of x

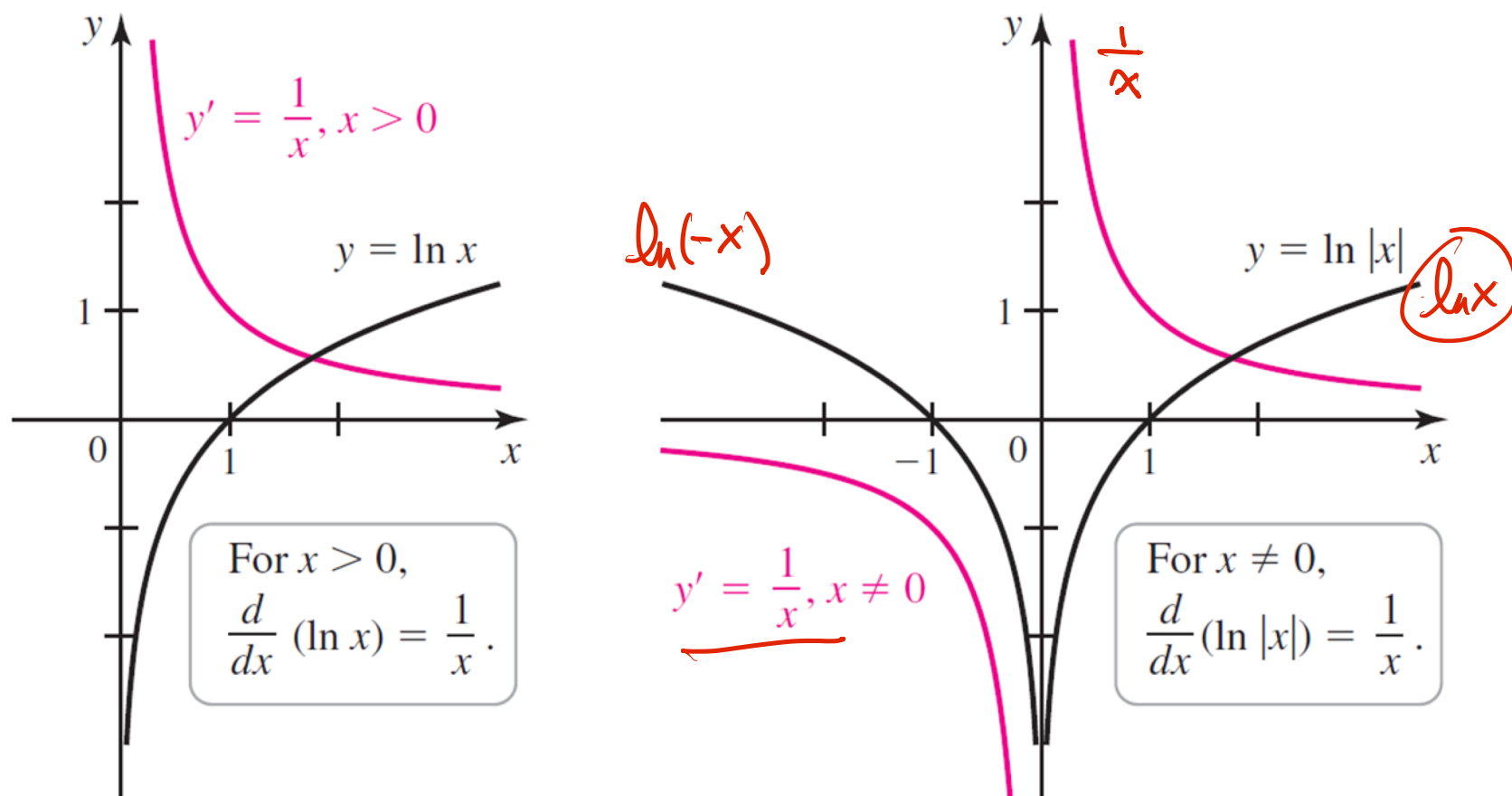
$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \text{ for } x > 0$$

$$\frac{d}{dx}(\ln |x|) = \frac{1}{x}, \text{ for } x \neq 0$$

If u is differentiable at x and $u(x) \neq 0$, then

$$\frac{d}{dx}(\ln |u(x)|) = \frac{u'(x)}{u(x)}.$$

Figure 3.63 (a & b)



Example 1 compute $y'(x)$

$$(a) \quad y = \ln 4x \\ = \ln 4 + \ln x$$

$$y' = 0 + \frac{1}{x}$$

$$(b) \quad y = x \ln x$$

$$y' = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

$$(c) \quad y = \ln |\sec x|$$

$$y' = \frac{1}{\sec x} \cdot (\sec x)' = \frac{\sec x \tan x}{\sec x}$$

$$= \tan x$$

$$(d) \quad y = \frac{\ln x^2}{x^2}, \quad y' = \frac{\frac{2x}{x^2} \cdot x^2 - 2x \cdot \ln x^2}{(x^2)^2}$$

$$= \frac{2 \ln x}{x^2}$$

$$y' = 2 \cdot \frac{\frac{1}{x} \cdot x^2 - \ln x \cdot 2x}{x^4} = 2 \cdot \frac{1 - 2 \ln x}{x^3}$$

Theorem 3.16 Derivative of b to the x Power

$$(e^x)' = \underline{e^x}$$

If $b > 0$ and $b \neq 1$, then for all x ,

$$y = b^x \quad \frac{d}{dx} \left[\ln y = \ln b^x = x \ln b \right] \quad \frac{d}{dx} (b^x) = b^x \ln b$$

$$\frac{1}{y} \cdot y' = \ln b \Rightarrow y' = y \ln b = b^x \ln b$$

$$(b^x)' = \left(e^{\ln b^x} \right)' = \left(e^{x \ln b} \right)' = e^{x \ln b} \cdot \ln b = b^x \ln b$$

Example

$$(1) f(t) = 3^t$$

$$f'(t) = 3^t \cdot \ln 3$$

$$(2) g(t) = 108 \cdot 2^{\frac{t}{12}}$$
$$g' = 108 \cdot 2^{\frac{t}{12}} \ln 2 \cdot \left(\frac{t}{12}\right)'$$
$$= 108 \cdot 2^{\frac{t}{12}} \ln 2 \cdot \frac{1}{12}$$

$$(3) y(x) = 5^{\sin x}$$

$$y' = 5^{\sin x} \cdot \ln 5 \cdot (\sin x)'$$
$$= 5^{\sin x} \ln 5 \cos x$$

Theorem 3.18 Derivative of Log Base b of x

If $b > 0$ and $b \neq 1$, then

$$\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}, \text{ for } x > 0 \text{ and } \frac{d}{dx}(\log_b |x|) = \frac{1}{x \ln b}, \text{ for } x \neq 0.$$

$$y = \log_b x$$

$$b^y = b^{\log_b x} = x$$

$$b^y \cdot \ln b \cdot y' = (x)' = 1$$

$$\Rightarrow y' = \frac{1}{b^y \ln b} = \frac{1}{x \ln b}$$

Example 4 $y'(x) = ?$

(a) $y = \underline{x^\pi}$; (b) $y = \underline{\pi^x}$; (c) $y = (x^2 + 4)^e$

(a) $y' = \pi \cdot x^{\pi-1}$

(b) $y' = \pi^x \ln \pi$

(c) $y' = e (x^2 + 4)^{e-1} \cdot (x^2 + 4)'$
 $= 2ex (x^2 + 4)^{e-1}$

• Logarithmic Differentiation

$$f(x) = \frac{(x^3-1)^4 \sqrt{3x-1}}{x^2+4}, \text{ compute } f'(x).$$

$$\ln f(x) = \ln \frac{(x^3-1)^4 \sqrt{3x-1}}{x^2+4} = 4 \ln(x^3-1) + \frac{1}{2} \ln(3x-1) - \ln(x^2+4)$$

$$(\text{LHS})' = (\ln f(x))' = f'(x)/f(x)$$

$$(\text{RHS})' = 4 \frac{(x^3-1)'}{x^3-1} + \frac{1}{2} \frac{(3x-1)'}{3x-1} - \frac{(x^2+4)'}{x^2+4} = \frac{4 \cdot 3x^2}{x^3-1} + \frac{3}{2(3x-1)} - \frac{2x}{x^2+4}$$

$$\Rightarrow f'(x) = (\text{RHS})' f(x) = f(x) \left[\frac{12x^2}{x^3-1} + \frac{3}{2(3x-1)} - \frac{2x}{x^2+4} \right]$$

82 $f(x) = \frac{x^8 \cos^3 x}{\sqrt{x-1}}, \text{ compute } f'(x).$

$$\left[\ln f(x) = \ln \frac{x^8 \cos^3 x}{\sqrt{x-1}} = \ln x^8 + \ln \cos^3 x - \ln \sqrt{x-1} \right]$$

$$// \quad = 8 \ln x + 3 \ln \cos x - \frac{1}{2} \ln(x-1)$$

$$\frac{f'(x)}{f(x)} = \frac{8}{x} + 3 \frac{(\cos x)'}{\cos x} - \frac{1}{2} \frac{(x-1)'}{x-1}$$

$$= \frac{8}{x} - \frac{3 \sin x}{\cos x} - \frac{1}{2} \frac{1}{x-1}$$

$$\Rightarrow f'(x) = \left[\frac{8}{x} - 3 \tan x - \frac{1}{2(x-1)} \right] f(x)$$

- $y = [f(x)]^{\underline{g(x)}}$, compute $\frac{dy}{dx}$

• power function vs exponential function

$$(x^n)' = n x^{n-1}$$

$$(b^x)' = b^x \cdot \ln b$$

$$\left(\underline{b^{f(x)}} \right)' = b^{f(x)} \ln b \cdot f'(x)$$

Example $y(x) = x^x$, $y'(x) = ?$

$$(x^x)' = \left[e^{\ln x^x} \right]' = \left[e^{x \ln x} \right]'$$

$$= e^{x \ln x} \cdot [x \ln x]' = x^x [\ln x + x \cdot (\ln x)']$$

$$= x^x [\ln x + 1]$$

$$\left[\underline{f(x)^{g(x)}} \right]' = \left[e^{\underline{g(x) \ln f(x)}} \right]' = f(x)^{g(x)} [g(x) \ln f(x)]'$$

$$= f(x)^{g(x)} \left[g'(x) \ln f(x) + \frac{g(x) \cdot f'(x)}{f(x)} \right]$$

Example

$$y = x^{\sin x}$$

$$\begin{aligned} \left(x^{\sin x} \right)' &= \left(e^{\frac{\sin x \ln x}{}} \right)' = x^{\sin x} \left[\sin x \ln x \right]' \\ &= x^{\sin x} \left[\cos x \ln x + \frac{\sin x}{x} \right] \end{aligned}$$

$$\bullet \left(f(x)^{g(x)} \right)' =$$

#107

$$\lim_{h \rightarrow 0} \frac{(3+h)^{3+h} - 27}{h} = f'(3) = 3^3 [\ln 3 + 1]$$

$$f(x) = x^x$$

$$(x^x)' = x^x \cdot [\ln x + 1]$$

$$f'(3) = \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \rightarrow 0} \frac{(3+h)^{3+h} - 3^3}{h}$$

$$\#108 \quad \lim_{x \rightarrow 2} \frac{5^x - 25}{x - 2} = \lim_{x \rightarrow 2} \frac{5^x - 5^2}{x - 2} = f'(2)$$

$$f(x) = 5^x, \quad f'(x) = 5^x \ln 5$$

$$f'(2) = 5^2 \ln 5 = 25 \ln 5$$

Gradelines

A: 75-100

D: 17-34

B: 54-73

F: 0-16

C: 35-53

#60. Determine whether the graph of $y = x^{\sqrt{x}}$ has any horizontal tangent lines.

$$\begin{aligned}
 0 &= (x^{\sqrt{x}})' = (e^{\sqrt{x} \ln x})' = x^{\sqrt{x}} [\sqrt{x} \ln x]' \\
 &= x^{\sqrt{x}} \left[\frac{1}{2} x^{\frac{1}{2}-1} \ln x + \sqrt{x} \cdot \frac{1}{x} \right] \\
 &= x^{\sqrt{x}} \left[\frac{1}{2} x^{-\frac{1}{2}} \ln x + x^{-\frac{1}{2}} \right] = x^{\sqrt{x}} \cdot x^{-\frac{1}{2}} \left[\frac{1}{2} \ln x + 1 \right] \\
 0 &= x^{\sqrt{x} - \frac{1}{2}} \left[\frac{1}{2} \ln x + 1 \right] \Leftrightarrow \begin{cases} x^{\sqrt{x} - \frac{1}{2}} = 0 ? \\ \text{or } \frac{1}{2} \ln x + 1 = 0 \\ \quad \boxed{x = e^{-2}} \quad \ln x = -2 \end{cases}
 \end{aligned}$$

#86 $f(x) = x^{(x^{10})}$

$$\begin{aligned}
 (x^{x^{10}})' &= \left[e^{\ln x^{x^{10}}} \right]' = \left[e^{\overbrace{x^{10} \ln x}^{\quad}} \right]' \\
 &= x^{x^{10}} \cdot [x^{10} \ln x]' \\
 &= x^{x^{10}} \cdot \left[(x^{10})' \ln x + x^{10} \cdot (\ln x)' \right] \\
 &= x^{x^{10}} \left[10 x^{10-1} \ln x + x^{10} \cdot \frac{1}{x} \right] = x^{x^{10}} [10 \ln x + 1] x^9 \\
 &= x^{x^{10} - 9} [10 \ln x + 1]
 \end{aligned}$$

$$A: 100 - 75$$

$$B: 73 - 54$$

$$C: 53 - 35$$

$$D: 34 - 17$$

$$F: 16 - 0$$

Section 3.10 Derivatives of Inverse Trigonometric Functions

(HW #1, 3, 5, 11, 13, 15, 18, 21, 30, 42, 45, 81, 89)

$$y = \tan^{-1} x = \frac{\sin^{-1} x}{\cos^{-1} x}$$

$$y = \tan x = \frac{\sin x}{\cos x}$$

$$x = \tan^{-1} y = \tan^{-1} \left(\frac{\sin x}{\cos x} \right)$$

$$\bullet \frac{d}{dx} (\sin^{-1} x) = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

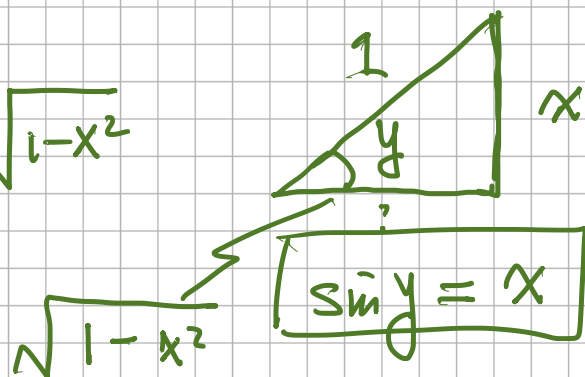
$$\sin^{-1} x \neq \frac{1}{\sin x}$$

$$y = \sin^{-1} x$$

$$[\sin y = x]'$$

$$\cos y \cdot y'(x) = 1 \Rightarrow y'(x) = \frac{1}{\cos y}$$

$$\cos y = ? \sqrt{1-x^2}$$



$$\bullet \frac{d}{dx} (\cos^{-1} x)$$

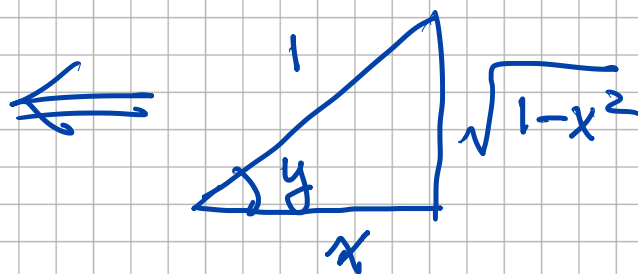
$$= -\frac{1}{\sqrt{1-x^2}}$$

$$y = \cos^{-1} x \Leftrightarrow$$

$$\cos y = x$$

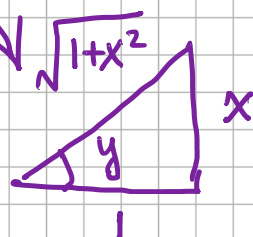
$$-\sin y \cdot y'(x) = 1 \Rightarrow y'(x) = -\frac{1}{\sin y}$$

$$\sin y = \sqrt{1-x^2}$$



$$y = \tan^{-1} x \Leftrightarrow [\tan y = x]' \Rightarrow \sec^2 y \cdot y'(x) = 1$$

$$y'(x) = \frac{1}{\sec^2 y} = \cos^2 y = \left(\frac{1}{\sqrt{1+x^2}} \right)^2 = \frac{1}{1+x^2}$$



$$\bullet \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\sec^2 y = \tan^2 y + 1 \\ = x^2 + 1$$

Examples

$$\#17 \quad f(x) = \sin^{-1}(e^{-2x})$$

$$f'(x) = \frac{1}{\sqrt{1-(e^{-2x})^2}} \cdot e^{(-2x)} \cdot (-2x)'$$

$$\#28 \quad f(t) = \ln(\sin^{-1} t^2)$$

$$f'(t) = \frac{1}{\sin^{-1} t^2} \cdot \frac{1}{\sqrt{1-(t^2)^2}} \cdot (t^2)'$$

• Derivatives of Inverse Functions in General

Let f be differentiable and have an inverse on an interval I .

Assume that $f'(x_0) \neq 0$

$\Rightarrow$ f^{-1} is differentiable at $y_0 = f(x_0)$ and $(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$

$$y = f(x) \iff \left[\underbrace{f^{-1}(y) = f^{-1}(f(x)) = x}_{f^{-1}(y(x))} \right]$$

$$(f^{-1})'(y) \cdot \underline{y'(x)} = 1 \quad \underline{f^{-1}(y(x))}$$

$$(f^{-1})'(y) = \frac{1}{y'(x)} = \frac{1}{f'(x)}$$

#12 Find the slope of the curve $y = \tan^{-1} x$ at $\left(\underline{1}, \underline{\frac{\pi}{4}}\right) = (x, y_0)$
without calculating the derivative of $\tan^{-1} x$.

$$[\tan y(x) = x]'$$

$$\sec^2 y(x) \cdot y'(x) = 1$$

$$\Rightarrow y'(x) = \frac{1}{\sec^2 y}$$

$$y'(1) = \cos^2 y(1) = \cos^2 \frac{\pi}{4} = \frac{1}{2}$$

$$y'(1) = \left. \frac{1}{1+x^2} \right|_{x=1} = \frac{1}{2}$$

Theorem 3.20 Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}, \text{ for } -1 < x < 1$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}, \text{ for } -\infty < x < \infty$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}, \text{ for } |x| > 1$$

Example 3 As the sun descends behind the 150-ft grandstand of a baseball stadium, the shadow of the stadium moves across the field. (a) express θ as a function of s . (b) compute $\frac{d\theta}{ds}$ when $s=200$.

Figure 3.72

$$\frac{d\theta}{ds} = \frac{d}{ds} \left(\tan^{-1} \frac{150}{s} \right) = \frac{1}{1 + \left(\frac{150}{s} \right)^2} \cdot \left(\frac{150}{s} \right)'$$

$$= \frac{1}{1 + \left(\frac{150}{s} \right)^2} \cdot 150 \cdot \left(-\frac{1}{s^2} \right)$$

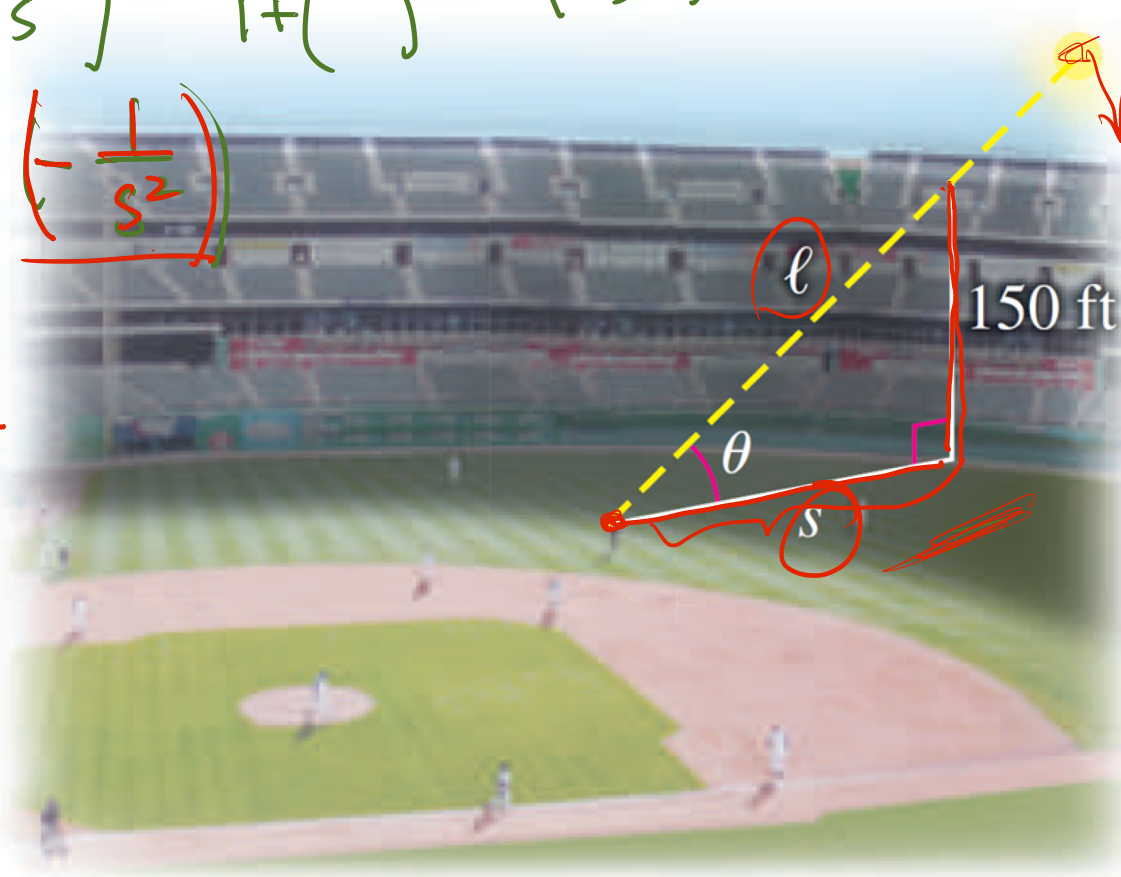
$$= - \frac{150}{s^2 + 150^2}$$

(b) compute $\frac{d\theta}{ds}$ when $s=200$.

$$\tan \theta = \frac{150}{s}$$

$$\theta = \tan^{-1} \left(\frac{150}{s(t)} \right)$$

$$\left(\frac{1}{s} \right)' = \left(s^{-1} \right)' = (-1) s^{-1-1}$$

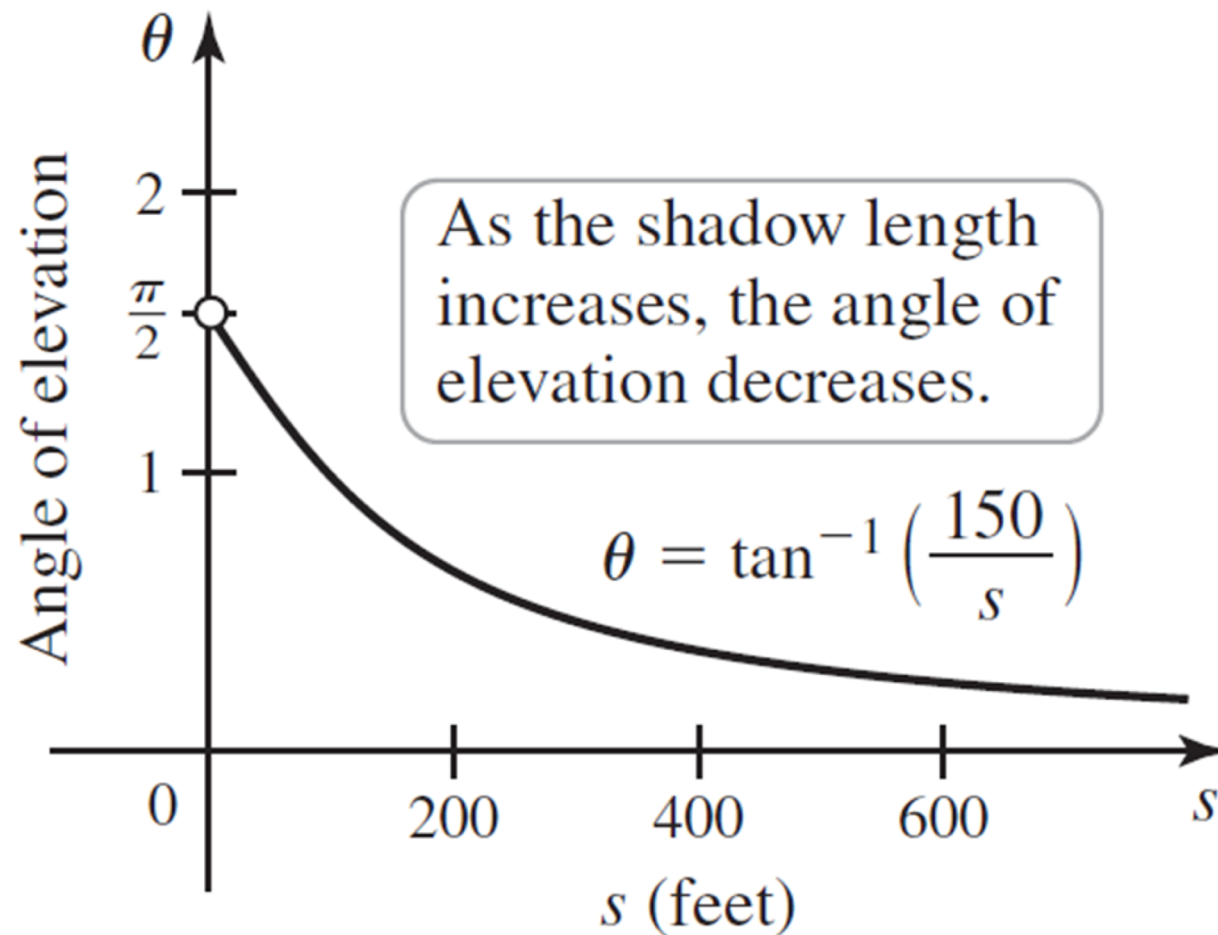


l : the line segment between the edge of the shadow and the sun

θ : the angle between l and the horizontal

s : the distance between the edge of the shadow and the base of the grandstand

Figure 3.73



Theorem 3.21 Derivative of the Inverse Function

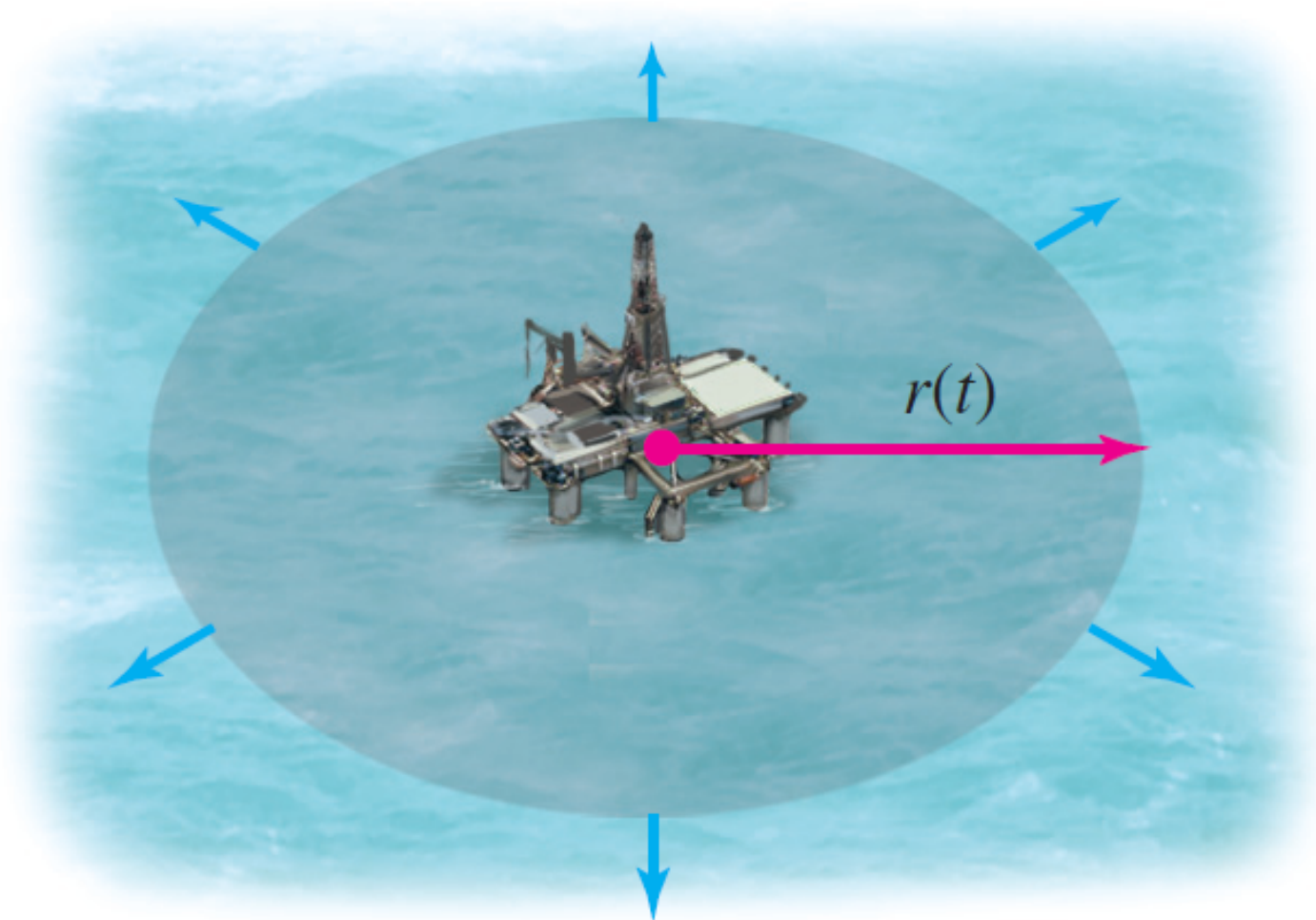
Let f be differentiable and have an inverse on an interval I . If x_0 is a point of I at which $f'(x_0) \neq 0$, then f^{-1} is differentiable at $y_0 = f(x_0)$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}, \quad \text{where } y_0 = f(x_0).$$

Section 3.11 Related Rates

(HW Part I #11, 13, 15, 19, 25, 29, 33, 35, 39 Part II #23, 24, 26, 117, 36, 41, 45, 47, 50, 61)

Figure 3.77



Procedure Steps for Related-Rate Problems

1. Read the problem carefully, making a sketch to organize the given information. Identify the rates that are given and the rate that is to be determined.
2. Write one or more equations that express the basic relationships among the variables.
3. Introduce rates of change by differentiating the appropriate equation(s) with respect to time t .
4. Substitute known values and solve for the desired quantity.
5. Check that units are consistent and the answer is reasonable. (For example, does it have the correct sign?)

Figure 3.78

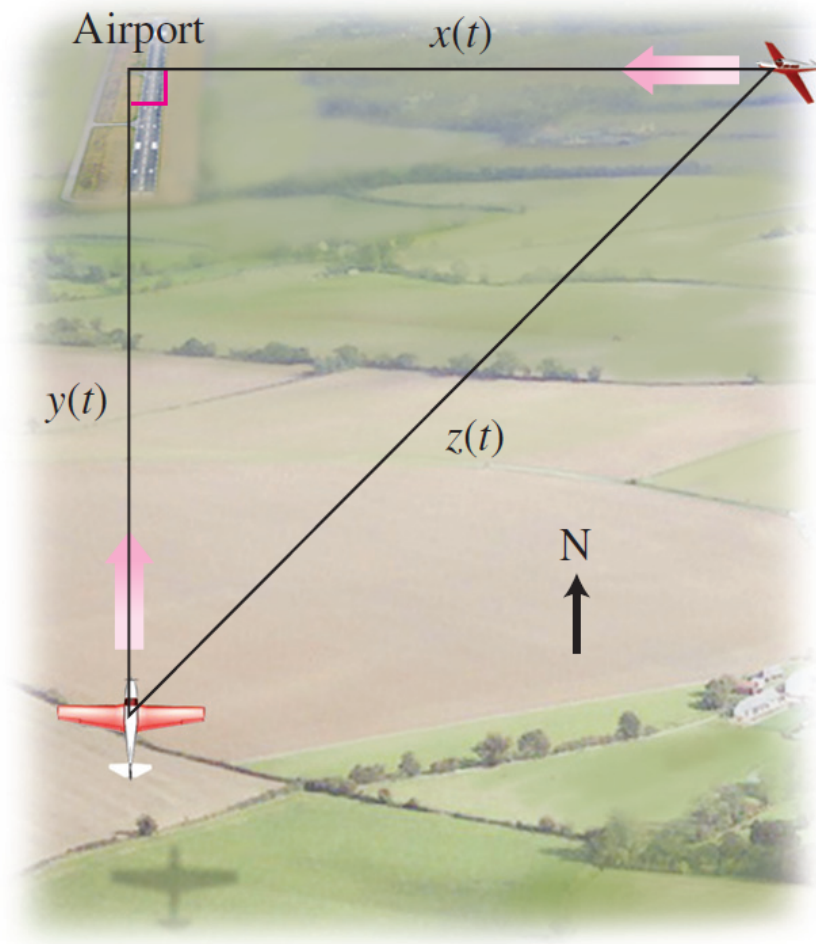


Figure 3.79 (a & b)

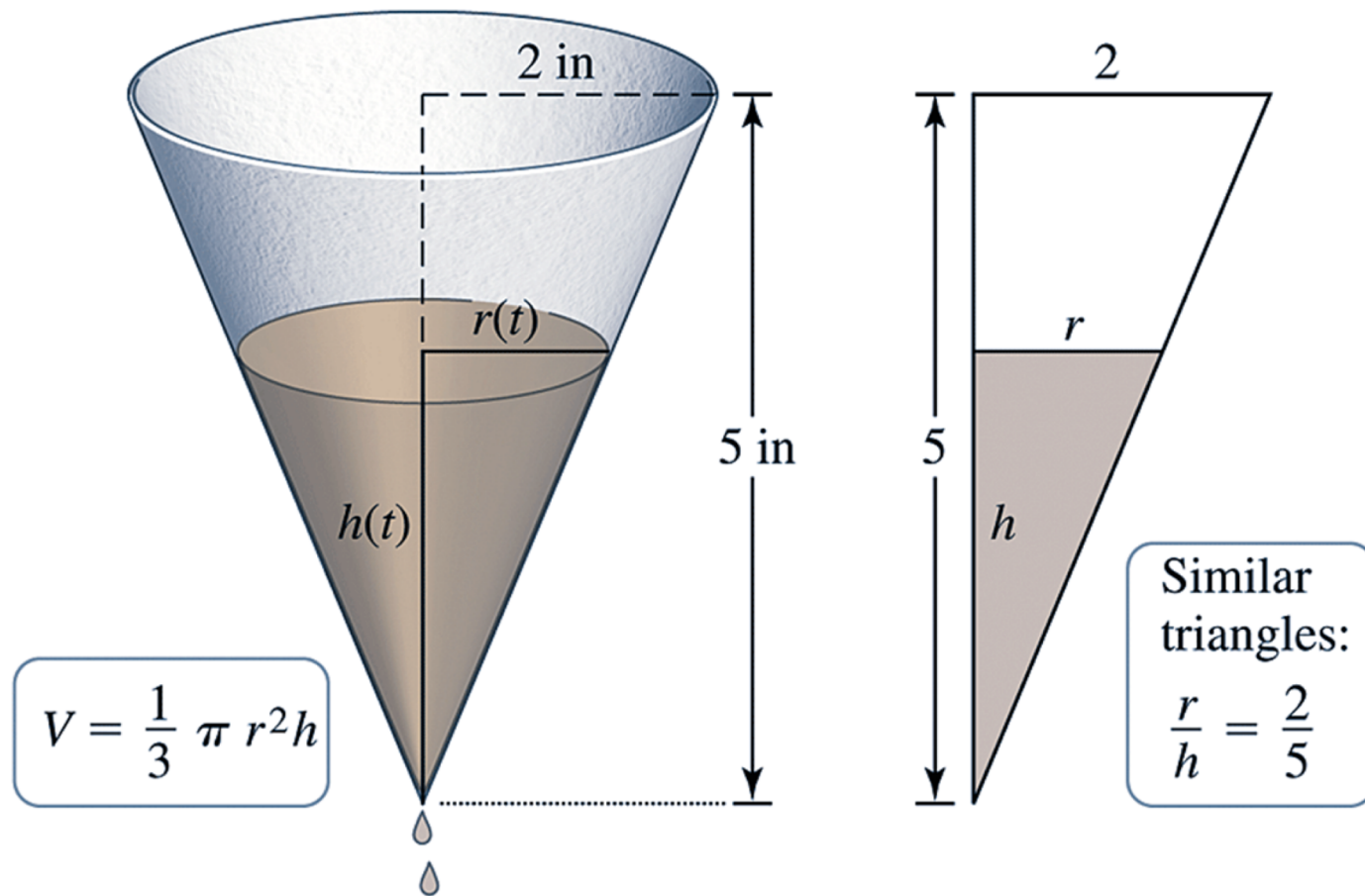


Figure 3.80

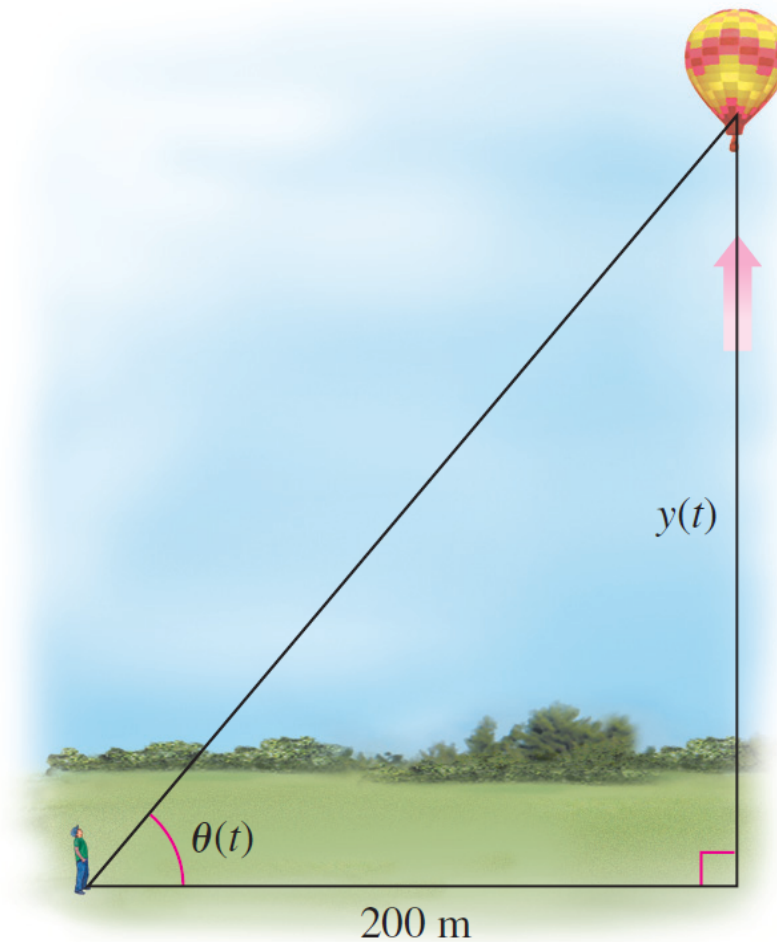
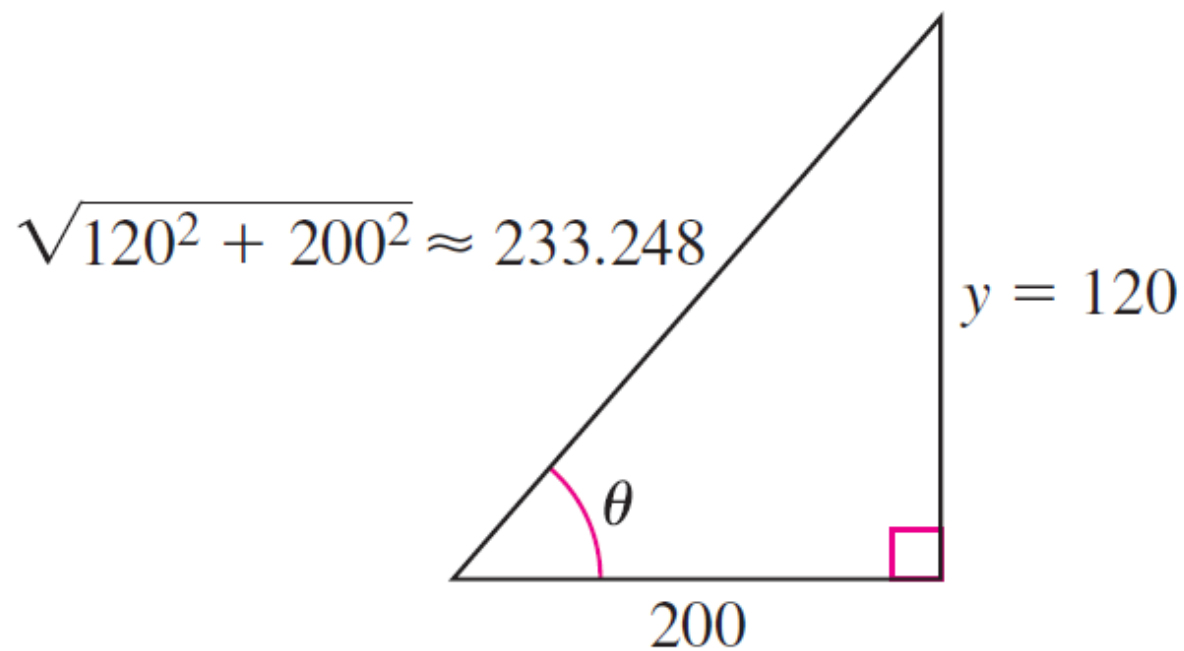


Figure 3.81



$$\cos \theta \approx \frac{200}{233.248} \approx 0.857$$